The University of Chicago

REPRESENTATION AS A SUM OF MULTIPLES OF POLYGONAL NUMBERS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
1936

By
MAE RUTH ANDERSON

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REPRESENTATION AS A SUM OF MULTIPLES OF POLYGONAL NUMBERS

INTRODUCTION

Dickson (1)[1] has proved that every positive integer A is the sum of k numbers, all but four of which are 0 or 1, of the type

$$p(x) = \tfrac{1}{2}m(x^2 - x) + x, \qquad x \geqq -1, \tag{1}$$

where k = 4, 5, 5, 6 if m = 3, 4, 5, 6 respectively, and where k = m - 1 if $m \geqq 7$. He has also proved in the same paper that when all but s of the values are 0 or 1 ($5 \leqq s \leqq k$), then k = m - 2 if m > 7 and k = s if $m \leqq 7$.

If m = 1, (1) gives the triangular numbers; if m = 2, the squares. Fermat stated that A is a sum of three triangular numbers and also a sum of four squares.

If $m \geqq 3$, (1) gives the polygonal numbers of order m + 2 and the extended polygonal number m - 1. We shall consider $m \geqq 3$ and study the representation of A by

$$f = a_1p_1 + a_2p_2 + \cdots + a_np_n$$

where $a_1, \ldots, a_n$ are fixed positive integers, $p_1, \ldots, p_n$ are numbers of the type (1), and the sum $a_1 + a_2 + \cdots + a_n \leqq$ the minimum value of k given by the theorems in the first paragraph of this section. The similar problem has been studied by Miss Griffiths (2, 3) in the cases where $p_1, \ldots, p_n$ are generalized,

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[1] See list of references at end of paper.

extended, and ordinary polygonal numbers.

We shall determine in Part I the necessary conditions on the coefficients of f so that f will represent 0, 1, ..., 15m + 6, and in Part II the lower limits for the integers represented by f when p_1, ..., p_4 are any polygonal numbers and p_5, ..., p_n are 0 or 1. Since these limits are lower when p_5 also may be any polygonal number, in Part III some new five-term Cauchy lemmas will be proved and in Part IV these and four others proved by Dickson (4) will be used to lower the limits for the integers represented by some of our functions f.

PART I

NECESSARY CONDITIONS ON THE COEFFICIENTS OF f

1. Polygonal Numbers of Orders 6, 7, 8. We shall use (1) and the notation

$$
\begin{aligned}
&f = a_1p_1 + \cdots + a_np_n = (a_1,\ldots,a_n), \quad 1 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n,\\
&f_k = (a_1,\ldots,a_k) \quad (1 \leqq k \leqq n-1),\\
&w_k = a_1 + \cdots + a_k \quad (1 \leqq k \leqq n-1),\\
&w = a_1 + \cdots + a_n,\\
&F_k = (a_k,\ldots,a_n) \quad (2 \leqq k \leqq n),\\
&w_j' = a_k + \cdots + a_j \quad (k \leqq j \leqq n).
\end{aligned} \tag{2}
$$

The first four values of (1) are $m - 1$, 0, 1, $m + 2$. Since $p(x + 1) - p(x) = mx + 1 > 1$ for $x > 0$, all integers are not polygonal numbers and hence $n \geqq 2$. In order that f shall represent 1, $a_1 = 1$.

If $m = 3$, (1,1), (1,2), (1,3), (1,1,1), and (1,1,2) do not represent 8, 8, 9, 20, and 43, respectively. Thus the only universal function is (1,1,1,1), the one given by Dickson. If $m \geqq 4$, note that $m - 1 \geqq 3$. Therefore, f represents 2 if and only if $a_2 = 1$ or 2. But $(1,2,a_3,\ldots,a_n)$, $n \geqq 2$, does not represent $m = (m - 1) + 1$ and (1,1) does not represent 3. Consequently $f = (1,1,a_3,\ldots,a_n)$, where $n \geqq 3$. If $m = 4$, (1,1,1), (1,1,2), (1,1,3), (1,1,1,1) do not represent 11, 25, 8, and 26 respectively; if $m = 5$, they do not represent 10, 17, 6, and 17 respectively; if $m = 6$, they do not represent 4, 33, 7, and 33 respectively. Hence if $m = 4, 5, 6$, the two remaining possibilities are

(1,1,1,1,1), which is Dickson's universal function, and (1,1,1,2).

2. Illustrative Lemmas for Polygonal Numbers of General Order. If $m \geqq 7$, we shall use $w \leqq m - 2$. Since, however, f does not represent $m - 2$ if $w < m - 2$, then $w = m - 2$. We shall now give a proof suggested by Dickson for a lemma used by Miss Griffiths (2, 3) in her papers.

LEMMA 1. The function $f = (1,1,a_3,\ldots,a_n)$ represents $0, 1, \ldots, w = m - 2$ if and only if $a_k \leqq w_{k-1} + 1$ $(3 \leqq k \leqq n)$.

If 3 is represented $a_3 = 1$, 2, or 3. Hence, for $k = 3$, the lemma holds. As a basis for the induction assume that if f_{k-1} represents $0, \ldots, w_{k-1}$, then $a_i \leqq w_{i-1} + 1$ $(3 \leqq i \leqq k - 1)$. We must prove that if f_k represents $0, \ldots, w_k$, then $a_k \leqq w_{k-1} + 1$. Assume $a_k > w_{k-1} + 1$. Since $w = m - 2$, f_{k-1} represents $0, 1, \ldots, w_{k-1}$ with no $p_i > 1$. Also $w_{k-1} + 1, \ldots, w_k$ must be represented in the same manner. The maximum value represented by f_{k-1} is w_{k-1}. Since $a_k > w_{k-1} + 1$, f_k does not represent $w_{k-1} + 1$, which is a contradiction.

Conversely, if $a_3 \leqq w_2 + 1$, that is, if $a_3 = 1$, 2, or 3, f_2 represents $0, 1, \ldots, w_3$. Assume that if $a_i \leqq w_{i-1} + 1$ $(2 \leqq i \leqq k - 1)$, then f_{k-1} represents $0, 1, \ldots, w_{k-1}$. Then f_{k-1} represents in particular $w_{k-1} - (a_k - 1)$, $w_{k-1} - (a_k - 1) + 1$, $\ldots, w_{k-1}$, since $w_{k-1} - (a_k - 1) \geqq 0$. Adding a_k to each of the terms, we find that f_k represents $w_{k-1} + 1, \ldots, w_{k-1} + a_k = w_k$.

The numbers $m - 1$ and m are represented by $f = (1,1,a_3,\ldots,a_n)$ but $m + 1 = (m - 1) + 2$ is represented if and only if $a_3 = 1$ or 2. We now prove

LEMMA 2. The function $f = (1,1,a_3,\ldots,a_n)$, $a_3 = 1$, 2, $a_k \leqq w_{k-1} + 1$ $(4 \leqq k \leqq n)$, represents $m + 2, m + 3, \ldots, (m + 2) + (w - 1) = 2m - 1$.

This is equivalent to saying that $F_2 = (1,a_3,\ldots,a_n)$

shall represent 0, 1, ..., w - 1, where $a_k \leqq w_{k-1}' + 2$ since $w_{k-1} = w_{k-1}' + 1$. The lemma then holds by Lemma 1 except when $a_k = w_{k-1}' + 2 = w_{k-1} + 1$. We shall now consider this case.

The first general a_k is a_4. When $a_3 = 1$, then $a_4 = w_{k-1} + 1 = 4$. Then, since $(m + 2) + 3 = (m - 1) + 6$, (1,1,1,4) obviously represents $m + 2, \ldots, (m + 2) + 6$. When $a_3 = 2$, $a_4 = w_{k-1} + 1 = 5$. Since $(m + 2) + 4 = (m - 1) + 7$, then (1,1,2,5) is seen to represent $m + 2, \ldots, (m + 2) + 8$.

Assume f_{k-1} $(k \geqq 5)$ represents $m + 2, \ldots, (m + 2) + (w_{k-1} - 1)$. By adding $a_k = w_{k-1} + 1$ to the above, we see that f_k represents the additional numbers $(m + 2) + (w_{k-1} + 1), \ldots, (m + 2) + (w_k - 1)$, with $(m + 2) + w_{k-1}$ missing. However, $(m + 2) + w_{k-1} = (m - 1) + w_k + 3 = (m - 1) + 2 + a_k$, which is represented.

Considering the interval 2m to 2m + 4, we find that $2m + 1 = (m + 2) + (m - 1)$ and $2m + 4 = 2(m + 2)$ are represented. But f with $a_3 = 1$ represents $2m = 2(m - 1) + 2$ only if $a_4 = 1, 2$, and f with $a_3 = 2$ represents $2m + 2 = (m + 2) + (m - 1) + 1 = 2(m - 1) + 4$ only if $a_4 = 2, 3, 4$. Then $2m + 3 = (m + 2) + (m - 1) + 2$ is also represented.

LEMMA 3. The function $f = (1,1,a_3,\ldots,a_n)$, $a_3 = 1$, $a_4 = 1, 2$; $a_3 = 2$, $a_4 = 2, 3, 4$; $a_k \leqq w_{k-1} + 1$ $(5 \leqq k \leqq n)$, represents $2(m + 2), \ldots, 2(m + 2) + (w - 2) = 3m$.

This is equivalent to saying that $F_2 = (a_3,\ldots,a_n)$ shall represent 0, ..., w - 2, where $a_k \leqq w_{k-1} + 1 = w_{k-1}' + 3$. Hence the lemma follows from Lemma 1 unless $a_k = w_{k-1}' + 2 = w_{k-1}$ or $a_k = w_{k-1}' + 3 = w_{k-1} + 1$, the cases which we shall now consider.

The first general a_k is a_5. Since $2(m + 2) + s = (m + 2) + (m - 1) + 3 + s$, then (1,1,1,1,4), (1,1,1,1,5), (1,1,1,2,5), and (1,1,1,2,6) represent $2(m + 2), \ldots, 2(m + 2) + (w_5 - 2)$

where $w_5 = 8, 9, 10$, and 11, respectively. Assume f_{k-1} $(k \geq 6)$ represents $2(m + 2), \ldots, 2(m + 2) + (w_{k-1} - 2)$. If $a_k = w_{k-1}$, then f_k evidently represents $2(m + 2) + w_{k-1}, \ldots, 2(m + 2) + (w_k - 2)$. The missing term $2(m + 2) + w_{k-1} - 1$ equals $(m + 2) + (m - 1) + 2 + a_k$, which is also represented. If $a_k = w_{k-1} + 1$, f_k obviously represents $2(m + 2) + w_{k-1} + 1, \ldots, 2(m + 2) + w_k - 2$ with the terms $2(m + 2) + w_{k-1} - 1$ and $2(m + 2) + w_{k-1}$ missing. But

$$2(m + 2) + w_{k-1} - 1 = (m + 2) + (m - 1) + 1 + a_k,$$

which is represented when $a_3 = 1$

$$= 2(m - 1) + 4 + a_k,$$

which is represented when $a_3 = 2$

and $2(m + 2) + w_{k-1} = (m + 2) + (m - 1) + 2 + a_k$, which is always represented.

3. Summary of Results for Polygonal Numbers of General Order. In order to determine further conditions on the coefficients, we chose arbitrarily to continue this process until we reached $p(6) = 15m + 6$. The intervals to which we applied the method of Lemmas 2 and 3 are: $3m + 3$ to $(3m + 3) + (w - 1) = 4m$; $(3m + 3) + (m + 2)$ to $(3m + 3) + (m + 2) + (w - 2) = 5m + 1$; $(3m + 3) + 2(m + 2)$ to $(3m + 3) + 2(m + 2) + (w - 3) = 6m + 2$; $6m + 4$ to $(6m + 4) + (w - 1) = 7m + 1$; $(6m + 4) + (m + 2)$ to $(6m + 4) + (m + 2) + (w - 2) = 8m + 2$; $(6m + 4) + 2(m + 2)$ to $(6m + 4) + 2(m + 2) + (w - 3) = 9m + 3$; $(6m + 4) + (3m + 3)$ to $(6m + 4) + (3m + 3) + (w - 2) = 10m + 3$; $10m + 5$ to $(10m + 5) + (w - 1) = 11m + 2$; $(10m + 5) + (m + 2)$ to $(10m + 5) + (m + 2) + (w - 2) = 12m + 3$; $2(6m + 4)$ to $2(6m + 4) + (w - 2) = 13m + 4$; $(10m + 5) + (3m + 3)$ to $(10m + 5) + (3m + 3) + (w - 2) = 14m + 4$; $(10m + 5) + (3m + 3) + (m + 2)$ to $(10m + 5) + (3m + 3) + (m + 2)$

+ (w - 3) = 15m + 5 for $(1,1,1,a_4,\ldots,a_n)$ and 2(6m + 4) + 2(m + 2) to 2(6m + 4) + 2(m + 2) + (w - 4) = 15m + 6 for $(1,1,2,a_4,\ldots,a_n)$. The numbers between the intervals were studied separately.

Without giving any more of the details we shall state the results. If $(1,1,1,1,a_5,\ldots,a_n)$ is to represent 3m + 2, then $n \geqq 5$, and if $a_5 = 3$ then $a_6 = 3, 4, 5$ and $n \geqq 6$; if $(1,1,2,3,a_5,\ldots,a_n)$ is to represent it, then $a_5 = 3, 4, 5$ and $n \geqq 5$. In order that 5m + 3 be represented, the following restrictions are imposed: in $(1,1,1,1,4,a_6,\ldots,a_n)$, $a_6 = 4, \ldots, 8$ and $n \geqq 6$; in $(1,1,1,1,5,a_6,\ldots,a_n)$, $a_6 = 5, \ldots, 8$ and $n \geqq 6$; in $(1,1,2,4,a_5,\ldots,a_n)$, $a_5 = 4, \ldots, 8$ and $n \geqq 5$. Also 5m + 5 is represented by $(1,1,1,2,a_5,\ldots,a_n)$ when $a_5 = 2, 3, 4$ only if $n \geqq 5$, when $a_5 = 5$ only if $a_6 = 5, \ldots, 10$ with $n \geqq 6$, and when $a_5 = 6$ only if some $a_h = 7$ or 10 $(h \geqq 7)$, with $n \geqq h$. If 5m + 6 is represented by $(1,1,2,2,a_5,\ldots,a_n)$ when $a_5 = 2, 3, 4, 5, 7$ then $n \geqq 5$, and when $a_5 = 6$ then some $a_h = 7, 8, 10, 11$ $(h \geqq 7)$ with $n \geqq h$. If $a_5 = 6$ or 7, then $(1,1,2,2,a_5,\ldots,a_n)$ does not represent $5m + 6 + a_5 + \cdots + a_n$. If $a_5 = 6$, then $(1,1,1,2,a_5\ldots,a_n)$ does not represent $5m + 17 + a_6 + \cdots + a_n$. If $a_7 \neq 14, 15, 16$ and if $n \geqq 7$, then $(1,1,1,1,5,6,a_7,\ldots,a_n)$ represents $9m + 11 + a_7 + \cdots + a_n$. For 11m + 12 to be represented by $(1,1,2,2,2,9,a_7,\ldots,a_n)$ when $a_7 = 10, \ldots, 18$ then $n \geqq 7$ and when $a_7 = 9$ then $a_8 = 9, \ldots, 23$ and $n \geqq 8$. If $a_7 = 17$ or 18, then $(1,1,2,2,2,9,a_7,\ldots,a_n)$ does not represent $11m + 12 + a_7 + \cdots + a_n$. Hence we have the following possibilities for f:

(3)
$$(1,1,1,1) \text{ for } m = 3;$$
$$(1,1,1,1,1) \text{ and } (1,1,1,2) \text{ for } m = 4, 5, 6.$$

$(1,1,1,1,a_5,\ldots,a_n)$ $\quad w = m - 2;\ a_k \leqq w_{k-1} + 1$

$a_5 = 1,\ 2;\ n \geqq 5,$ $\quad (6 \leqq k \leqq n),$

$a_5 = 3;\ a_6 = 3,\ 4,\ 5;\ n \geqq 6$ $\quad (7 \leqq k \leqq n),$

(4) $\quad a_5 = 4;\ a_6 = 4,\ \ldots,\ 8;\ n \geqq 6$ $\quad (7 \leqq k \leqq n),$

$a_5 = 5;\ a_6 = 5,\ 7,\ 8;\ n \geqq 6$ $\quad (7 \leqq k \leqq n),$

$a_5 = 5;\ a_6 = 6;\ a_7 \neq 14,\ 15,\ 16;\ n \geqq 7$ $\quad (8 \leqq k \leqq n),$

$(1,1,1,2,a_5,\ldots,a_n)$ $\quad w = m - 2;\ a_k \leqq w_{k-1} + 1$

(5) $\quad a_5 = 2,\ 3,\ 4;\ n \geqq 5$ $\quad (6 \leqq k \leqq n),$

$a_5 = 5;\ a_6 = 5,\ \ldots,\ 10;\ n \geqq 6$ $\quad (7 \leqq k \leqq n),$

$(1,1,2,2,a_5,\ldots,a_n)$ $\quad w = m - 2;\ a_k \leqq w_{k-1} + 1$

$a_5 = 2;\ a_6 = 2,\ \ldots,\ 8;\ n \geqq 6$ $\quad (7 \leqq k \leqq n),$

(6) $\quad a_5 = 2;\ a_6 = 9;\ a_7 = 9;\ a_8 = 9,\ \ldots,\ 23;\ n \geqq 8$ $\quad (9 \leqq k \leqq n),$

$a_5 = 2;\ a_6 = 9;\ a_7 = 10,\ \ldots,\ 16;\ n \geqq 7$ $\quad (8 \leqq k \leqq n),$

$a_5 = 3,\ 4,\ 5;\ n \geqq 5$ $\quad (6 \leqq k \leqq n),$

(7) $\quad (1,1,2,3,a_5,\ldots,a_n)$ $\quad w = m - 2;\ a_k \leqq w_{k-1} + 1$

$a_5 = 3,\ 4,\ 5;\ n \geqq 5$ $\quad (6 \leqq k \leqq n),$

(8) $\quad (1,1,2,4,a_5,\ldots,a_n)$ $\quad w = m - 2;\ a_k \leqq w_{k-1} + 1$

$a_5 = 4,\ \ldots,\ 8;\ n \geqq 5$ $\quad (6 \leqq k \leqq n).$

PART II

DETERMINATION OF LOWER LIMITS FOR UNIVERSAL FUNCTIONS

4. General Formula. We shall determine the lower limit for the positive integers A which are represented by (2_1) with the restrictions (3_3), (4), ..., (8). In order to do this we first limit ourselves to the case where $p_1, \ldots, p_4$ are any polygonal numbers (1), but $p_5, \ldots, p_n$ are either 0 or 1. Denote the excess $a_5p_5 + \cdots + a_np_n$ by r. Then

$$A = \tfrac{1}{2}ma + \tfrac{1}{2}nb + r, \quad a = \sum_{i=1}^{4} a_i x_i^2, \quad b = \sum_{i=1}^{4} a_i x_i, \tag{9}$$
$$n = 2 - m, \quad 0 \leqq r \leqq E = a_5 + \cdots + a_n, \quad t = a_1 + \cdots + a_4.$$

Dickson (4) proved the following result:

LEMMA 4. Let $(a_1,\ldots,a_4)$ be one of the five sets below. Let

$$a, b \text{ of (9) be integers}, \quad b \geqq -t, \quad a \equiv b \pmod 2, \tag{10}$$
$$ta \geqq b^2, \qquad (t-1)a < b^2 + 4b + 4t.$$

Then there exist integral solutions > -2 of a and b if the following further conditions hold:

(1,1,1,1), $4a - b^2 \neq 4^m(8n + 7)$.

(1,1,1,2), $4(5a - b^2) \neq 25^m(5n \pm 2)$; true if either a or b is prime to 5.

(1,1,2,2), $6a \cdot b^2 \neq 4^m(8n + 7)$; true if b is odd.

(1,1,2,3), $6(7a - b^2) \neq 49^m(7n + e)$, $e = 3, 5, 6$; true if either a or b is prime to 7.

(1,1,2,4), b odd or double an odd.

In his discussion of representation as a sum of polygonal numbers, Dickson (5) replaced the limits on a and b by a limit on A, involving m. The method we use is similar to his.

Multiplying (10_5) by m and substituting for a from (9_1) gives

$$mb^2 + 4bm + 4tm > 2(t - 1)A - (t - 1)nb - 2(t - 1)r.$$

Since this inequality follows from the one obtained from it by suppressing r, we shall use the latter. Completing the square on b gives

$$(2mb + h)^2 > U, \quad h = 4m + (t - 1)n, \tag{11}$$
$$U = 8(t - 1)mA - 16tm^2 + h^2,$$

where (11_1) is implied by

$$b > (U^{1/2} - h)/2m \quad \text{if } U \geqq 0. \tag{12}$$

We multiply (10_4) by m, substitute for a from (9_1), complete the square on b, and notice that the inequality is implied by the one obtained by replacing r by E, that is, by

$$(2mb + nt)^2 \leqq V \quad \text{where} \quad V = 8tm(A - E) + n^2t^2. \tag{13}$$

If $2mb + nt \geqq 0$ and $V \geqq 0$, (13) follows from

$$b \leqq (V^{1/2} - nt)/2m. \tag{14}$$

But $2mb + nt \geqq 0$ if n and b are $\geqq 0$. If $n < 0$, $2mb + nt \geqq 0$ by (12) if

$$U^{1/2} \geqq h - nt \tag{15}$$

and hence if

(16) $$8(t - 1)mA > 16tm^2 - 2hnt + n^2t^2.$$

We want $b > 0$, which is true by (12) if $U^{1/2} \geqq h$. If $n < 0$, this follows from (15). If $n \geqq 0$, then $h \geqq 0$ and $U^{1/2} \geqq h$ if and only if $U \geqq h^2$, and hence if

(17) $$(t - 1)A \geqq 2tm.$$

There will be at least d integers between the values of b given by (12) and (14) if

$$(V^{1/2} - nt)/2m - (U^{1/2} - h)/2m > d$$

and hence if

(18) $$V^{1/2} - U^{1/2} > P \quad \text{where} \quad P = 2md + n - 4m.$$

The left member $\geqq 0$ if $V \geqq U$. If also $P \geqq 0$, (18) holds if its square holds and hence if

(19) $$F \equiv (V + U - P^2)^2 - 4UV \geqq 0, \qquad P \geqq 0.$$

We shall call $U \geqq 0$, $V \geqq 0$, $V \geqq U$, and (16) or (17) the minor conditions. But $V \geqq U$ if

(20) $$A \geqq tE,$$

which also implies $V \geqq 0$. Also $U \geqq 0$ if

(21) $$(t - 1)A \geqq 2tm.$$

Hence, if $n \geqq 0$, the minor conditions all follow from (20) and (21); if $n < 0$, from (16), (20), and (21).

5. Special Cases. We consider first f satisfying (3). Since (1,1,1,1) for $m = 3$ and (1,1,1,1,1) for $m = 4, 5, 6$ are universal by Dickson's theorem, we need investigate only

LEMMA 5. Let $f = (1,1,1,2)$, $m = 4, 5, 6$, and let β be any integer, $0 \leqq \beta \leqq A$. Then there exist integers a and b, each $\geqq 0$, such that $A = m(a - b)/2 + b + r$, $a \equiv b \pmod 2$, a and b not both divisible by 5, and b is one of $\beta, \ldots, \beta + 2m - 1$.

We may write $A - \beta = mg + \rho$, where $\rho = 0, 1, \ldots, m - 1$. A suitable choice of a and b must be made so that A can be written in the form (9_1) with $r = 0$; i.e. $A = m(a - b)/2 + b$. Let $b = \beta + \rho$, $a = 2g + b$ unless $a \equiv b \equiv 0 \pmod 5$. Then we write $A = m(g - 1) + \beta + m + \rho$, and let $b = \beta + m + \rho$, $a = 2(g - 1) + b$, where $b \not\equiv 0 \pmod 5$ if $m = 4, 6, 7$ and $a \not\equiv 0 \pmod 5$ if $m = 5$. Hence the largest value of b is $\beta + m + m - 1 = \beta + 2m - 1$. Thus $d = 2m$.

When $m = 4$, then $d = 8$, $t = 5$, $n = -2$, $E = 0$. Then (11), (13), (18), and (19) give

$$U = 64(2A - 19), \quad V = 4(40A + 25), \quad P = 46,$$
$$U + V - P^2 = 32(9A - 101), \quad F = 1024(A^2 - 1108A + 10676).$$

Thus $F \geqq 0$ if $A \geqq 1108$. This value of A also satisfies the minor conditions. Hence we have the first part of Theorem 1.

When $m = 5$, we have $d = 10$, $t = 5$, $n = -3$, $E = 0$. Then

$$U = 16(10A - 121), \quad V = 25(8A + 9), \quad P = 77,$$
$$U + V - P^2 = 40(9A - 191), \quad F = 1600(A^2 - 2560A + 37570).$$

Hence $F \geqq 0$ if $A \geqq 2560$. In this case all the minor conditions are also satisfied.

For $m = 6$, we have $d = 12$, $t = 5$, $n = -4$, $E = 0$. Hence

$$U = 64(3A - 44), \quad V = 16(15A + 25), \quad P = 116,$$
$$U + V - P^2 = 16(27A - 992), \quad F = 2304(A^2 - 4912A + 111296).$$

Therefore, $F \geqq 0$ if $A \geqq 4912$, a value of A which also satisfies

the minor conditions.

6. First Function. For f satisfying (4), ..., (8), we need another lemma also due to Miss Griffiths (2, 3).

LEMMA 6. Let $f = (a_1,\ldots,a_n)$ as in (2) with the understanding that $w_0 = 0$. If $a_k \leqq w_{k-1} + 1$, $a_i = w_{i-1} + s$ $(s \geqq 2)$, $(1 \leqq k \neq i \leqq n)$, then f represents 0, 1, ..., w, except $w_{i-1} + 1$, ..., $w_{i-1} + s - 1$, and perhaps $w_{i-1} + 1 + A_1$, ..., $w_{i-1} + s - 1 + A_1$ where A_1 is the sum of one or more of the a_j following a_i.

By (2), $F_5 = (a_5,\ldots,a_n)$; $w_{k-1}' = a_5 + \cdots + a_{k-1}$. By (9), $t = a_1 + \cdots + a_4$. Hence $w_{k-1} = t + w_{k-1}'$. Since, by hypthesis, $a_k \leqq w_{k-1} + 1$, we have

$$a_k \leqq w_{k-1}' + 1 + t. \tag{22}$$

First, we take $f = (1,1,1,1,a_5,\ldots,a_n)$. Since the method fails if a_5, ..., a_n are all divisible by 4, we assume that

(23) among a_5, ..., a_n there is a first coefficient a_h not divisible by 4.

We also note that if a is odd or double an odd, the condition in Lemma 4_1 is satisfied; that is, $4a - b^2 \neq 4^m(8n + 7)$.

LEMMA[1] 7. Let $f = (1,1,1,1,a_5,\ldots,a_n)$, with $a_k \leqq w_{k-1} + 1$ $(5 \leqq k \leqq n)$, be subject to (23) and let β be any odd integer, $a_h - 1 \leqq \beta \leqq A$. Then there exist integers a, b, r, each $\geqq 0$, such that $A = m(a - b)/2 + b + r$, $a \equiv b \pmod 2$, $a \not\equiv 0 \pmod 4$, F_5 represents $r \leqq m - 6$, and b is one of $\beta - a_h + 1$, ..., $\beta + a_h + 5$.

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[1] Lemmas 8 (with the restriction that β be odd) and 7 are given without proof in Miss Griffiths' second paper. Lemma 10, which is a revision of another lemma, appears in a manuscript of hers, and we are are indebted to her for a portion of its proof.

By (22) any a_k in $F_5 = (a_5,\ldots,a_n)$ satisfies the condition $a_k \leqq w_{k-1}' + 5$. By Lemma 6, if $a_k = w_{k-1}' + s$, the only numbers not represented by F_5 are $w_{k-1}' + 1 + B_k$, ..., $w_{k-1}' + s - 1 + B_k$ where $B_k = 0$ and where B_k may perhaps also equal A_k. In either case F_5 represents $w_{k-1}' + B_k$ which can also be written as $(w_{k-1}' + 1 + B_k) - 1$, $(w_{k-1}' + 2 + B_k) - 2$, ..., $(w_{k-1}' + s - 1 + B_k) - (s - 1)$. Denote by R any number not represented by F_5. Since $s = 2, 3, 4, 5$, we see that one of $R - 1$, $R - 2$, ..., $R - 4$ is represented.

We may write $A = mg + \beta + \rho$ where β is odd and $0 \leqq \rho \leqq m - 1$. If ρ is represented by F_5, put $r = \rho$, $b = \beta$, $a = 2g + b$. If ρ is not represented by F_5, then $0 < \rho < m - 6$ or $\rho = m - 1, m - 2, \ldots, m - 5$. If $0 < \rho < m - 6$ and $\rho - s$ is represented where $s = 2$ or 4, put $r = \rho - s$, $b = \beta + s$, $a = 2g + b$. If $\rho - s$ is represented where $s = 1$ or 3, do the same unless $a \equiv 0 \pmod 4$. But $\rho - s + a_h$ or $\rho - s - a_h$ is also represented, since $\rho - s$ is represented with $p_h = 0$ or 1. If $\rho - s + a_h$ is represented, put $r = \rho - s + a_h$, $b = \beta + s - a_h$, $a = 2g + b$; if $\rho - s - a_h$ is, put $r = \rho - s - a_h$, $b = \beta + s + a_h$, $a = 2g + b$. In either case, $a \not\equiv 0 \pmod 4$. If $\rho = m - s$ $(s = 1, \ldots, 5)$, then $\rho - (6 - s)$ is represented. If s is even, put $r = \rho - (6 - s)$, $b = \beta + (6 - s)$, $a = 2g + b$. If s is odd do likewise unless $a \equiv 0 \pmod 4$. In that case, since $\rho - (6 - s + a_h)$ is also represented, put $r = \rho - (6 - s + a_h)$, $b = \beta + (6 - s + a_h)$, $a = 2g + b \not\equiv 0 \pmod 4$.

Evaluating (11), (13), (18), and (19) with $t = 4$, $d = 2a_h + 6$, $E = m - 6$, $n = 2 - m$, we have

$$U = 24mA - 63m^2 + 12m + 36, \quad P = 2 + m(4a_h + 7),$$
$$V = 32mA - 16m^2 + 128m + 64, \quad U + V - P^2 = 8(7mA - jm^2 + sm + 12),$$
$$j = 16 + 7a_h + 2a_h^2, \quad s = 14 - 2a_h,$$

$$F/64 = m^2A^2 - (74 + 98a_h + 28a_h^2)m^3A - (20 + 28a_h)m^2A + cm^4 + fm^3 + gm^2 + hm,$$

$$c = 193 + 224a_h + 113a_h^2 + 28a_h^3 + 4a_h^4,$$

$$f = 68 - 132a_h - 28a_h^2 + 8a_h^3,$$

$$g = 4 - 224a_h - 44a_h^2, \quad h = -48a_h.$$

Since $m > 3$ it is evident that the sum of the last four terms $\geqq 0$. Hence

$$F \geqq 0 \text{ if } A \geqq (74 + 98a_h + 28a_h^2)m + 20 + 28a_h.$$

The minor conditions require a smaller A. Hence if

$$a_5 = 1, \quad A \geqq 200m + 48,$$

$$a_5 = 2, \quad A \geqq 382m + 76,$$

$$a_5 = 3, \quad A \geqq 620m + 104,$$

$$a_5 = 5, \quad A \geqq 1264m + 160.$$

7. Second Function.

LEMMA 8. Let $f = (1,1,1,2,a_5,\ldots,a_n)$, where $a_k \leqq w_{k-1} + 1$ $(5 \leqq k \leqq n)$ and where

(24) there is a first coefficient a_h not divisible by 5 among $a_5, \ldots, a_n$.

Let β be any integer, $a_h \leqq \beta \leqq A$. Then there exist integers a, b, r, each $\geqq 0$, such that $A = m(a - b)/2 + b + r$, $a \equiv b \pmod 2$, a or b not divisible by 5, F_5 represents $r \leqq m - 7$, and b is one of $\beta - a_h, \ldots, \beta + a_h + 6$.

We may write $A = mg + \beta + \rho$ where β is any integer, $a_h \leqq \beta \leqq A$. If F_5 represents ρ, put $r = \rho$, $b = \beta$, $a = 2g + b$ unless $a \equiv b \equiv 0 \pmod 5$. Then, since either $\rho + a_h$ or $\rho - a_h$ is also represented, put $r = \rho \pm a_h$ as the case may be, $b = \beta$

$\mp a_h \not\equiv 0 \pmod 5$, $a = 2g + b$. Since $a_k \leqq w_{k-1}' + 6$, if F_5 does not represent ρ it represents $\rho - s$, where $s = 1, \ldots,$ or 5. If ρ is not represented when $0 < \rho < m - 7$, let $r = \rho - s$, $b = \beta + s$, $a = 2g + b$ unless $a \equiv b \equiv 0 \pmod 5$. In that case, let $r = \rho - s \pm a_h$, $b = \beta + s \mp a_h \not\equiv 0 \pmod 5$, $a = 2g + b$, the sign being chosen so that F_5 represents r. If $\rho = m - s$ ($s = 1, \ldots, 6$), let $r = \rho - (7 - s)$, $b = \beta + (7 - s)$, $a = 2g + b$ unless $a \equiv b \equiv 0 \pmod 5$, when we let $r = \rho - (7 - s + a_h)$, $b = \beta + (7 - s + a_h) \not\equiv 0 \pmod 5$, $a = 2g + b$.

Here $d = 2a_h + 7$, $t = 5$, $E = m - 7$. Hence

$$U = 32mA - 80m^2 + 64, \qquad V = 40mA - 15m^2 + 180m + 100,$$

$$P = 2 + m(9 + 4a_h), \qquad U + V - P^2 = 8(9mA - jm^2 + sm + 20),$$

$$j = 22 + 9a_h + 2a_h^2, \qquad s = 18 - 2a_h,$$

$$F/64 = m^2A^2 - (166 + 162a_h + 36a_h^2)m^3A - 36(1 + a_h)m^2A + cm^4 + fm^3 + gm^2 + hm,$$

$$c = 409 + 396a_h + 165a_h^2 + 36a_h^3 + 4a_h^4,$$

$$f = 108 + 236a_h - 36a_h^2 + 8a_h^3,$$

$$g = 4 - 432a_h - 76a_h^2, \qquad h = -80a_h.$$

Then $F \geqq 0$ if $A \geqq (166 + 162a_h + 36a_h^2)m + 36(1 + a_h)$. This value of A also satisfies the minor conditions. Hence if

$$a_5 = 2, \quad A \geqq 634m + 108,$$

$$a_5 = 3, \quad A \geqq 976m + 144,$$

$$a_5 = 4, \quad A \geqq 1390m + 180.$$

8. Third Function. In order to apply Lemma 4 to $f = (1,1,2,2,a_5,\ldots,a_n)$, we must have $6a - b^2 \neq 4^m(8n + 7)$. The lemma states that this is true if a and b are odd. If a is even and the highest power of 2 which it contains is even, i.e., $a =$

$a = 2^{2h}A$, $b = 2^{i}B$, $A \equiv B \equiv 1 \pmod 2$, $i, h > 0$, the condition is not fulfilled if $h > i$. For, then

$$6a - b^2 = 2^{2i}\left\{2^{2(h-i)+1}3A - B^2\right\} = 4^m(8n + 7),$$

since $2(h - i) + 1 \geqq 3$. But if $h \leqq i$,

$$6a - b^2 = 2^{2h}\left\{6A - 2^{2(i-h)}B^2\right\} \neq 4^m(8n + 7).$$

If $h < i$, this is obviously true. If $h = i$, $6a - b^2 = 4^m(8n + 7)$ would imply $6A \equiv 0 \pmod 8$. Similarly we consider the case where the highest power of 2 contained in A is odd, and hence prove that $6a - b^2 \neq 4^m(8n + 7)$ if

(25)
$$\begin{aligned} &a = 2^{2h}A,\ b = 2^{i}B, \text{ and } h \leqq i,\\ &a = 2^{2h-1}A,\ b = 2^{i}B, \text{ and } h = i + 1,\ h = i,\ h = i - 1 \text{ with}\\ &\qquad A \not\equiv 1 \pmod 8, \text{ or } h < i - 1 \text{ with } A \not\equiv 5 \pmod 8,\\ &\text{where } A \equiv B \equiv 1 \pmod 2 \text{ and } h, i > 0. \end{aligned}$$

Then $6a - b^2 = 4^m(8n + 7)$ if

(26)
$$\begin{aligned} &a = 2^{2h}A,\ b = 2^{i}B,\ h > i,\\ &a = 2^{2h-1}A,\ b = 2^{i}B, \text{ and } h > i + 1,\ h = i - 1 \text{ with}\\ &\qquad A \equiv 1 \pmod 8, \text{ or } h < i - 1 \text{ with } A \equiv 5 \pmod 8,\\ &\text{where } A \equiv B \equiv 1 \pmod 2, \text{ and } h, i > 0. \end{aligned}$$

LEMMA 9. *If* a *and* b *satisfy* (26), *then* $a \pm 4$, $b \pm 4$ *satisfy* (25). *If* a *and* b *satisfy* (26) *excepting* (26_1) *and* (26_2), *both with* $i = 1$, *then* $a + 2$ *and* $b + 2$ *satisfy* (25), *while if* a *and* b *satisfy the exceptional cases, then* $a \pm 8$, $b \pm 8$ *and* $a - 6$, $b - 8$ *satisfy* (25). *If* a *and* b *satisfy* (26) *excepting* (26_4) *with* $i = 1$, *then* $a - 2$ *and* $b - 2$ *satisfy* (25) *while if* a *and* b *satisfy* (26_4) *with* $i = 1$, *then* $a - 8$ *and* $b - 8$ *satisfy* (25).

We shall prove the first statement for a and b satisfying (26_1). Then $a \pm 4 = 2^2\left\{2^{2(h-1)>0}A \pm 1\right\}$; $b \pm 4 = 2\left\{B \pm 2\right\}$ if

$i = 1$, $2^{(2+s)} \equiv 3 \pmod{B}$ if $i = 2$, and $2^2(2^{i-2}B \pm 1)$ if $i \geqq 2$. Hence we have (25_1) with $h' = 1$, $i' = 1, 2, 2 + s$. The other proofs are made in a similar manner.

LEMMA 10. Let $f = (1,1,2,2,a_5,\ldots,a_n)$, $a_5 = 2, 3, 4, 5$, with $a_k \leqq w_{k-1} + 1$ $(6 \leqq k \leqq n)$, and let β be any odd integer, where $a_5 + a_6 - 1 \leqq \beta \leqq A$ when $a_5 = 2$, $a_6 = 2$, $a_6 - 1 \leqq \beta \leqq A$ when $a_5 = 2$, $a_6 \neq 2$, and $a_5 - 1 \leqq \beta \leqq A$ otherwise. Then there exist integers a, b, r, each $\geqq 0$ such that $A = m(a - b)/2 + b + r$, $a \equiv b \pmod 2$, $6a - b^2 \neq 4^m(8n + 7)$, F_5 represents $r \leqq m - 8$, and b is one of $\beta + 1 - (a_5 + a_6), \ldots, \beta + 5 + (a_5 + a_6)$ when $a_5 = 2$, $a_6 = 2$, one of $\beta + 1 - a_6, \ldots, \beta + 7 + a_5$ when $a_5 = 2$, $a_6 = 3$, one of $\beta + 1 - a_6, \ldots, \beta + 5 + a_6$ when $a_5 = 2$, $a_6 = 4, \ldots, 9$, one of $\beta + 1 - a_5, \ldots, \beta + 7 + a_5$ when $a_5 = 3, 4, 5$.

If F_5 represents ρ put $r = \rho$, $b = \beta$, $a = 2g + b$. Since $a_k \leqq w_{k-1}' + 7$, if F_5 does not represent ρ, $0 < \rho < m - 8$, F_5 does represent one of $\rho - 1, \ldots, \rho - 6$. This is also true if $\rho = (m - 2), \ldots, m - 7$. If F_5 represents $\rho - s$ where s is odd, put $r = \rho - s$, $r = \beta + s$, $a = 2g + b$. If s is even, do likewise unless $6a - b^2 = 4^m(8n + 7)$. To consider the latter situation together with the case $\rho = m - 1$, we subdivide the cases.

If $a_5 = 3, 4, 5$ we note that either $\rho - s - a_5$ or $\rho - s + a_5$ is also represented. Hence, let $r = \rho - s \pm a_5$, the sign being chosen so that r is represented, $b = \beta + s \mp a_5$, $a = 2g + b$. If a_5 is odd, b is odd; if $a_5 = 4$, a and b now satisfy (25) by Lemma 9. If $\rho = m - 1$, put $r = \rho - 7$, $b = \beta + 7$, $a = 2g + b$ unless a and b satisfy (26). Then let $r = \rho - 7 - a_5$, $b = \beta + 7 + a_5$, $a = 2g + b$.

If $a_5 = 2$, consider $\rho = m - 1$ first. Put $r = \rho - 7$, $b = \beta + 7$, $a = 2g + b$ unless a and b satisfy (26). Then put

$r = \rho - 7 - a_5$, $b = \beta + 7 + a_5$, $a = 2g + b$ unless the same situation occurs again. In that case, note that $A = mg + \beta + \rho = m(g + 1) + \beta - 1$. If $r = 0$, $b = \beta - 1 = (\beta + 7) - 8$, $a = 2(g + 1) + b = (2g + \beta + 7) - 6$, then (25) is now satisfied by Lemma 9. The discussion for the case where F_5 represents $\rho - s$, s even, is finished as in the preceding paragraph with a_6 substituted for a_5 when $a_6 = 3, 4, 5, 7, 9$. If $a_6 = 2, 6, 8$, we note that $\rho - s - a_5$ is represented except when $\rho = 1$, a case we shall consider separately. Hence put $r = \rho - s - a_5$, $b = \beta + s + a_5$, $a = 2g + b$, unless again $6a - b^2 = 4^m(8n + 7)$. Then use the fact that a_6 is either included in the representation of $\rho - s$ and consequently of $\rho - s - a_5$ or it is not. If $a_6 = 2$ and $\rho - s + a_6$ is represented, put $r = \rho - s + a_6$, $b = \beta + s - a_6 = (\beta + s + a_5) - 4$, $a = 2g + b$. If $\rho - s - a_6$ is represented, so is $\rho - s - a_5 - a_6$. Put $r = \rho - s - a_5 - a_6$, $b = \beta + s + a_5 + a_6 = (\beta + s) + 4$, $a = 2g + b$. If $\rho = 1$, put $r = 0$, $b = \beta + 1$, $a = 2g + b$, unless a and b satisfy (26). Then put $r = (\rho - 1) + a_5 + a_6$, $b = \beta + 1 - a_5 - a_6$, $a = 2g + b$. If $a_6 = 6$ and $\rho - s - a_5 + a_6$ is represented, put $r = \rho - s - a_5 + a_6$, $b = (\beta + s) + (a_5 - a_6) = (\beta + s) - 4$, $a = 2g + b$. If $\rho - s - a_6$ is represented, put $r = \rho - s - a_6$, $b = \beta + s + a_6 = (\beta + s + a_5) + 4$, $a = 2g + b$. If $\rho = 1$, put $r = \rho + 1 = a_5$, $b = \beta - 1$, $a = 2g + b$, unless a and b satisfy (26). In that case let $r = \rho + 5 = a_6$, $b = \beta - 5 = (\beta - 1) - 4$, $a = 2g + b$. If $a_6 = 8$, since one of $\rho - s \pm a_6$ is represented, put $r = \rho - s \pm a_6$, $b = \beta + s \mp a_6$, $a = 2g + b$, where (25) is now satisfied by Lemma 9. If $\rho = 1$, put $r = \rho - 1 = 0$, $b = \beta + 1$, $a = 2g + b$, unless a and b satisfy (26). Then $r = \rho + 1 = a_5$, $b = \beta - 1$, $a = 2g + b$ or $r = \rho + 7 = a_6$, $b = \beta - 7 = (\beta - 1) - 6$, $a = 2g + b$ gives an a and b which satisfy (25).

If $a_5 = 2$, $a_6 = 3$, then $\beta + 7 + a_5 > \beta + 5 + a_6$, but if $a_6 \neq 3$ the inequality is reversed. Hence $d = 14$ if $a_5 = 2$, $a_6 = 2$; $d = 13$, if $a_5 = 2$, $a_6 = 3$; $d = 2a_6 + 6$ if $a_5 = 2$, $a_6 > 3$; and $d = 2a_5 + 8$ if $a_5 = 3, 4, 5$.

Since d has several values, we evaluate (11), (13), (18), and (19) for a general d, but with $t = 6$, $E = m - 8$, obtaining

$$U = 40mA - 95m^2 - 20m + 100, \qquad V = 48mA - 12m^2 + 240m + 144,$$

$$P = 2 + m(2d - 5), \qquad U + V - P^2 = 4(22mA - jm^2 + sm + 60)$$

$$j = 33 - 5d + d^2, \qquad s = 60 - 2d,$$

$$F/16 = 4m^2A^2 - (192 - 220d + 44d^2)m^3A - (88d - 1480)m^2A + cm^4 + fm^3 + gm^2 + hm,$$

$$c = 804 - 330d + 91d^2 - 10d^3 + d^4,$$

$$f = 1680 + 732d - 140d^2 + 4d^3,$$

$$g = 4560 + 360d - 116d^2, \qquad h = 1920 - 240d.$$

By examination of the values of d, we see that $F \geqq 0$ if $A \geqq (48 - 55d + 11d^2)m + (22d - 370)$. This value of A also satisfies the minor conditions. Hence if

$$a_5 = 3, \quad A \geqq 1434m - 62,$$

$$a_5 = 4, \quad A \geqq 1984m - 18,$$

$$a_5 = 5, \quad A \geqq 2622m + 26.$$

9. Fourth Function.

LEMMA 11. Let $f = (1,1,2,3,a_5,\ldots,a_n)$, where $a_k \leqq w_{k-1} + 1$ $(5 \leqq k \leqq n)$ satisfy (7) and let β be any integer, $a_5 \leqq \beta \leqq A$. Then there exist integers a, b, r, each $\geqq 0$, such that $A = m(a - b)/2 + b + r$, $a \equiv b \pmod 2$, a or b not divisible by 7, F_5 represents $r \leqq m - 9$, and b is one of $\beta - a_5, \ldots, \beta + 8 + a_5$.

Write $A = mg + \beta + \rho$. If ρ is represented by F_5, put $r = \rho$, $b = \beta$, $a = 2g + b$ unless $a \equiv b \equiv 0 \pmod 7$. In that

case let $r = \rho \pm a_5$, the sign being chosen so that F_5 represents r, $b = \beta \mp a_5$, $a = 2g + b$, where now $b \not\equiv 0 \pmod 7$ since $a_5 = 3, 4, 5$. Since $a_k \leqq w_{k-1}' + 8$, if ρ is not represented by F_5, $0 < \rho < m - 9$, $\rho - s$ is represented where $s = 1, 2, \ldots,$ or 7. Let $r = \rho - s$, $b = \beta + s$, $a = 2g + b$ unless $a \equiv b \equiv 0 \pmod 7$. In that case, let $r = \rho - s \pm a_5$, the sign being chosen so that r is represented, $b = \beta + s \mp a_5$, $a = 2g + b$. If $\rho = m - s$ $(s = 1, \ldots, 8)$, let $r = \rho - (9 - s)$, $b = \beta + (9 - s)$, $a = 2g + b$ unless $a \equiv b \equiv 0 \pmod 7$. Then we let $r = \rho - (9 - s + a_5)$, $b = \beta + (9 - s + a_5)$, $a = 2g + b$.

Hence $d = 2a_5 + 9$, $t = 7$, $E = m - 9$. Thus

$$U = 48mA - 108m^2 - 48m + 144, \qquad V = 56mA - 7m^2 + 308m + 196,$$

$$P = 2 + m(4a_5 + 13), \qquad U + V - P^2 = 4(26mA - jm^2 + sm + 84),$$

$$j = 71 + 26a_5 + 4a_5^2, \qquad s = 52 - 4a_5,$$

$$F/16 = 4m^2A^2 - 4(524 + 338a_5 + 52a_5^2)m^3A - 4(80 + 52a_5)m^2A + cm^4 + fm^3 + gm^2 + hm,$$

$$c = 4852 + 3692a_5 + 1244a_5^2 + 208a_5^3 + 16a_5^4,$$

$$f = 848 - 2136a_5 - 208a_5^2 + 32a_5^3,$$

$$g = 16 - 4784a_5 - 656a_5^2, \qquad h = -672a_5.$$

Then $F \geqq 0$ if $A \geqq (524 + 338a_5 + 52a_5^2)m + 80 + 52a_5$. This value of A satisfies all the minor conditions. Hence, if

$$a_5 = 3, \quad A \geqq 2006m + 236$$

$$a_5 = 4, \quad A \geqq 2708m + 288$$

$$a_5 = 5, \quad A \geqq 3514m + 340.$$

10. Fifth Function.

LEMMA 12. Let $f = (1,1,2,4,a_5,\ldots,a_n)$, where $a_k \leqq w_{k-1} + 1$ $(5 \leqq k \leqq n)$, and where

(27) *among* $a_5, \ldots, a_n$ *there is a first coefficient* a_h *which is not divisible by* 4.

Let β *be any odd integer,* $a_h - 1 \leqq \beta \leqq A$. *Then there exist integers* a, b, r, *each* $\geqq 0$, *such that* $A = m(a - b)/2 + b + r$, $a \equiv b \pmod 2$, $b \not\equiv 0 \pmod 4$, F_5 *represents* $r \leqq m - 10$, *and* b *is one of* $\beta - a_h + 1, \ldots, \beta + a_h + 9$.

As before, $A = mg + \beta + \rho$, where now β is odd. If ρ is represented by F_5, let $r = \rho$, $b = \beta$, $a = 2g + b$. If ρ is not represented by F_5, $0 < \rho < m - 10$, then $\rho - s$ is represented where $s = 1, \ldots,$ or 8 since $a_k \leqq w_{k-1}' + 9$. If s is even, let $r = \rho - s$, $b = \beta + s$. If s is odd, do the same unless $b \equiv 0 \pmod 4$. In that case, let $r = \rho - s \pm a_h$, $b = \beta + s \mp a_h \not\equiv 0 \pmod 4$, $a = 2g + b$, the sign being chosen so that r is represented. Also F_5 does not represent $\rho = m - s$ $(s = 1, \ldots, 9)$. If s is even, let $r = \rho - (10 - s)$, $b = \beta + (10 - s)$, $a = 2g + b$. If s is odd, do the same unless $b \equiv 0 \pmod 4$. Then let $r = \rho - (10 - s + a_h)$, $b = \beta + (10 - s + a_h)$, $a = 2g + b$.

Therefore, $d = 2a_h + 10$, $t = 8$, $E = m - 10$. Hence,

$$U = 56mA - 119m^2 - 84m + 196, \qquad V = 64mA + 384m + 256,$$

$$P = 2 + m(4a_h + 15), \qquad U + V - P^2 = 8(15mA - jm^2 + sm + 56),$$

$$j = 43 + 15a_h + 2a_h^2, \qquad s = 30 - 2a_h,$$

$$F/64 = m^2A^2 - (814 + 450a_h + 60a_h^2)m^3A - (108 + 60a_h)m^2A + cm^4 + fm^3 + gm^2 + hm,$$

$$c = 1849 + 1290a_h + 397a_h^2 + 60a_h^3 + 4a_h^4,$$

$$f = 276 - 718a_h - 60a_h^2 + 8a_h^3,$$

$$g = 4 - 1800a_h - 220a_h^2, \qquad h = -224a_h.$$

Then $F \geqq 0$ if $A \geqq (814 + 450a_h + 60a_h^2)m + 108 + 60a_h$. In this case the minor conditions are satisfied. Hence if

$$a_5 = 5, \quad A \geqq 4564m + 408,$$
$$a_5 = 6, \quad A \geqq 5674m + 468,$$
$$a_5 = 7, \quad A \geqq 6904m + 528.$$

Thus we have proved

THEOREM 1. Let $f_1 = (1,1,1,2)$ satisfy (3_3), $f_2 = (1,1,1,1,a_5,\ldots,a_n)$ satisfy (4) and (23), $f_3 = (1,1,1,2,a_5,\ldots,a_n)$ satisfy (5) and (24), $f_4 = (1,1,2,2,a_5,\ldots,a_n)$ satisfy (6), $f_5 = (1,1,2,3,a_5,\ldots,a_n)$ satisfy (7), and $f_6 = (1,1,2,4,a_5,\ldots,a_n)$ satisfy (8) and (27). Then $f_1,\ldots,f_6$ are universal except for integers A such that $A < N$ for f_1 and $15m < A < N$ otherwise, where

$$N = 1108; 2560; 4912 \text{ for } f_1,\ m = 4, 5, 6,$$
$$N = 2m(37 + 49a_h + 14a_h^2) + 4(5 + 7a_h) \text{ for } f_2,$$
$$N = 2m(83 + 81a_h + 18a_h^2) + 36(1 + a_h) \text{ for } f_3,$$
$$N = m(48 - 55d + 11d^2) + 2(11d - 185) \text{ for } f_4,$$
$$N = 2m(262 + 169a_5 + 26a_5^2) + 4(20 + 13a_5) \text{ for } f_5,$$
$$N = 2m(407 + 225a_h + 30a_h^2) + 12(9 + 5a_h) \text{ for } f_6.$$

PART III

CAUCHY LEMMAS

11. Introduction and Summary. An investigation of the lower limit for A in Theorem 1 shows that the smallest value is 200m + 48 which occurs in the case (1,1,1,1,...). In order to prove that the function also represents A where $0 < A < 200m + 48$ it would be necessary to construct a table of sums of four of our numbers to cover that interval. To construct even this table would be quite a task. However, the four-term lemmas which Dickson (4) proved have about half as long an interval for b^2 as four five-term lemmas, which appear in the same paper, have. In this section we shall prove some new five-term lemmas which we shall apply later, together with those given by Dickson, in order that we may lower the limit for A. We shall also prove some six-term lemmas which do not offer any particular advantage over the others in this problem.

We investigate the conditions under which there exist integral solutions $> -k$ of the pair of equations

$$a = \sum_{i=1}^{s} c_i x_i^2, \quad b = \sum_{i=1}^{s} c_i x_i, \tag{28}$$

where k is an integer and the c_i are positive integers whose sum is denoted by t. Necessary conditions are

$$a, b \text{ integers}, \quad b \geqq t(1 - k), \quad a \equiv b \pmod 2. \tag{29}$$

We shall prove

THEOREM 2. Let $(c_1,\ldots,c_5)$ be one of the seven sets

below. Let (29) hold and

$$(30) \qquad ta \geqq b^2, \qquad (t-1)a < b^2 + 2bk + tk^2.$$

Then there exist integral solutions $> -k$ of (28) with $s = 5$ if the following further conditions hold:

$(1,1,1,1,4)$, b odd.
$(1,1,1,1,5)$.
$(1,1,1,2,4)$, $9a - b^2 \neq 2$.
$(1,1,2,2,2)$, b odd.
$(1,1,2,2,4)$.
$(1,1,2,2,5)$, $11a - b^2 \neq 2, 6, 8$.
$(1,1,2,4,4)$, b odd, $12a - b^2 \neq 3$.

THEOREM 3. Let $(c_1, \ldots, c_5)$ be one of the five sets below. Equations (28) with $s = 5$ have integral solutions $\geqq 0$ if $b \geqq 0$, $a \equiv b \pmod 2$,

$$ta \geqq b^2, \qquad (t-2)(a-2) < b^2,$$

and the following further conditions hold:

$(1,1,1,1,4)$, b odd.
$(1,1,1,1,5)$, b odd, $3(8a - b^2) \neq 25^m(25n \pm 10)$,
b even, $3\{8(a-1) - (b-1)^2\} \neq 25^m(25n \pm 10)$.
$(1,1,1,2,4)$, $9a - b^2 \neq 2$.
$(1,1,2,2,2)$, b odd, $7a - b^2 \neq 4^m(16n + 14)$.
$(1,1,2,2,4)$, b even; b odd, $9a - b^2 \neq 4^k(8n + 7)$.

THEOREM 4. Let $(c_1, \ldots, c_6)$ be one of the seven sets below. Let $a \equiv b \pmod 2$, $b \geqq 0$, and

$$(t-1)a \geqq b^2 + e, \qquad (t-2)(a-2) < b^2.$$

Then there exist integral solutions $\geqq 0$ of (28) with $s = 6$ if the following further conditions hold:

$$(1,1,1,1,1,4), \quad e = 1.$$
$$(1,1,1,1,1,5), \quad e = 1.$$
$$(1,1,1,2,2,2), \quad e = 1.$$
$$(1,1,1,2,2,4), \quad e = 2.$$
$$(1,1,1,1,2,4), \quad e = 3.$$
$$(1,1,1,2,2,5), \quad e = 9.$$
$$(1,1,1,2,4,4), \quad e = 9.$$

THEOREM 5. For the five sets in Theorem 3, equations (28) with $s = 5$ have integral solutions $> -k$ if we assume (29) and

$$ta \geqq b^2, \ (t-2)(a-2) < b^2 + 4b(k-1) + 2t(k-1)^2.$$

THEOREM 6. For the seven sets in Theorem 4, equations (28) with $s = 6$ have integral solutions $> -k$ if we assume (29) and

$$(t-1)a \geqq b^2 + e, \ (t-2)(a-2) < b^2 + 4b(k-1) + 2t(k-1)^2$$

12. Proof of Theorem 2. In the proof, we shall use a fundamental identity discovered by Dickson (4, p. 515). Denote $c_1 + \cdots + c_5$ by t_1 (the t of Theorem 2). Define t_2, t_3 by

$$t_k = t_{k-1} - c_{k-1}$$

and F_1, F_2, F_3 by

$$F_k = (t_k - c_k)x_k - \sum_{j=k+1}^{5} c_j x_j.$$

Then we have

$$F_k = F_{k+1} + (t_k - c_k)(x_k - x_{k+1}),$$

and

$$(t_1 - c_1)(t_2 - c_2)(t_3 - c_3)(t_1 a - b^2) \equiv (t_2 - c_2)(t_3 - c_3)c_1 P^2 + (t_3 - c_3)t_1 c_2 d^2 + t_1 t_2 c_3 R^2 + t_1 t_2 t_3 c_4 c_5 W^2, \tag{31}$$

where P, d, R replace F_1, F_2, F_3 and $W = x_4 - x_5$. Solving the linear equation with the left members b, P, d, R, W, we have

$$\begin{aligned} t_1 x_1 &= b + P, \qquad t_1 t_2 x_2 = t_2 b - c_1 P + t_1 d, \\ t_1 t_2 t_3 x_3 &= t_2 t_3 b - c_1 t_3 P - c_2 t_1 d + t_1 t_2 R, \\ (c_4 + c_5) t_1 t_2 t_3 x_4 &= G + c_5 t_1 t_2 t_3 W, \\ (c_4 + c_5) t_1 t_2 t_3 x_5 &= G - c_4 t_1 t_2 t_3 W, \end{aligned} \tag{32}$$

where

$$G = (t_3 - c_3)(t_2 t_3 b - c_1 t_3 P - c_2 t_1 d) - c_3 t_1 t_2 R.$$

Case (1,1,1,1,4). Let $c_1 = 4$. Cancel 8 from (31) and get

$$3(8a - b^2) = 3P^2 + 2d^2 + 4R^2 + 12W^2.$$

Make the substitution $d = R + 3D$ where $D = x_2 - x_3$. Hence

$$f = P^2 + 2T^2 + 4D^2 + 4W^2 \text{ shall represent } M = 8a - b^2,$$

where $T = R + D$. Then $d = T + 2D$, $R = T - D$, and (32) gives

$$\begin{aligned} 8x_1 &= b + P, \quad 8x_2, \quad 8x_3 = b - P + 2T \pm 4D, \\ &8x_4, \quad 8x_5 = b - P - 2T \pm 4W. \end{aligned}$$

Since the method fails when b is even, we assume b odd. Then $-b^2 \equiv P^2$, and $P \equiv b \pmod 2$. From $-b^2 \equiv b^2 + 2T^2 \pmod 4$, we get $T \equiv b \pmod 2$. By choice of the sign of P, $P = 4U - b$. Then

$$8a - b^2 \equiv 8Ub + b^2 + 2b^2 + 4D^2 + 4W^2 \pmod{16},$$

$$1 \equiv 2U + D^2 + W^2 \pmod 4.$$

Either D or W is odd. By interchange of D and W if necessary let D be odd and W even. Then $U \equiv 0 \pmod 2$.

Hence the x_1 are all integers. They are all $> -k$ if $-P$, $P - 2T \mp 4D$, $P + 2T \mp 4W$ are all $< b + 8k$. Dickson (4, p. 520) proved

LEMMA 13. *If* $L = d_1y_1 + \cdots + d_4y_4 \geqq 0$ *and each* $d_i \geqq 0$, *then*

$$L \leqq \left\{(d_1 + d_2 + d_3 + d_4)(d_1y_1^2 + d_2y_2^2 + d_3y_3^2 + d_4y_4^2)\right\}^{\frac{1}{2}}.$$

By this lemma, $|P| \leqq \sqrt{f}$, $|P| + 2|T| + 4|D| \leqq \sqrt{7f}$, and $|P| + 2|T| + 4|W| \leqq \sqrt{7f}$. The x_1 are then all $> -k$ if $\sqrt{7f} < b + 8k$, which gives (30_2).

We shall prove that for $a \equiv b \equiv 1 \pmod 2$ and for M positive, M is represented by f. Put $W = 0$. Dickson (6) proved that the ternary (1, 2, 4) represents all positive integers not of the form $4^k(16n + 14)$. Since M is odd, M is represented.

Case (1,1,1,1,5). Let $c_1 = 5$. Cancel 6 from (31) and get

$$4(9a - b^2) = 5P^2 + 3d^2 + 6R^2 + 18W^2.$$

Make the substitution $d = R + 3D$ where $D = x_3 - x_4$. Hence

$$f = 5P^2 + 9T^2 + 18D^2 + 18W^2 \text{ shall represent } M = 4(9a - b^2),$$

where $T = R + D$. Then $R = T - D$, $d = T + 2D$, and (32) gives

$$9x_1 = b + P, \quad 36x_2, \quad 36x_3 = 4b - 5P + 9T \pm 18D,$$
$$36x_4, \quad 36x_5 = 4b - 5P - 9T \pm 18W.$$

Since $b^2 \equiv P^2 \pmod 9$, by choice of the sign of P we have $P \equiv -b \pmod 9$.

Since $0 \equiv P^2 + T^2 \pmod 2$, $P \equiv T \pmod 2$. Then $0 \equiv 2T^2 + 2D^2 + 2W^2 \pmod 4$. Hence $T + D + W$ is even. First, let T

be even. Then $T = 2t$, $P = 2p$, and $D \equiv W \pmod 2$. Substituting, $0 \equiv 4p^2 + 4t^2 + 4D^2 \pmod 8$, whence $p + t + D$ is even. If $p \equiv t \pmod 2$, D and W are even. If $p \equiv D \equiv W \pmod 2$, t is even. If $t \equiv D \pmod 2$, p is even. Second, let T be odd. By interchange of D and W if necessary, let D be odd and W even. By choice of the sign of T we have $P \equiv -T \pmod 4$.

Hence the x_i are all integers. They are all $> -k$ if $-4P$, $5P - 9T \mp 18D$, $5P + 9T \mp 18W$ are all $< 4(b + 9k)$. By Lemma 13, $4|P|$, $5|P| + 9|T| + 18|D|$, $5|P| + 9|T| + 18|W|$ are all $\leqq \sqrt{32f}$. Then the x_i are all $> -k$ if $\sqrt{32f} < 4(b + 9k)$. This gives (30_2).

We show that for $a \equiv b \pmod 2$ and $M = 4(9a - b^2)$ positive, M is represented by $5P^2 + 3d^2 + 6R^2 + 18W^2$. Put $N = M - 5P^2$. Choose P odd and $\equiv b \pmod 3$. Then $N \equiv 0 \pmod 3$. Jones (7, p. 72) proved that the ternary (1, 2, 6) represents all positive integers $\neq 4^k(8n + 3)$. Since $N \equiv 3 \pmod 8$ and $N \equiv 0 \pmod 3$, it follows that $N = 8s + 3 \equiv 0 \pmod 3$. Then $N = 24\ell + 3$ and $N/3 = 8\ell + 1 \neq 4^k(8n + 5)$. Since P was chosen modulo 6, $|P| < 3$. Then N is negative if $M < 5P^2 < 45$. Since $M \equiv 0 \pmod 8$, the only values of $M < 45$ are 8, 16, 24, 32, 40. Of these, 8, 24, 32 are all represented by f. But $M = 16$, 40 would imply $9a - b^2 = 4$, 10 whence $b^2 \equiv 5, 8 \pmod 9$, which is impossible.

Case (1,1,1,2,4). Let $c_1 = 4$, $c_3 = 2$. Cancel 2 from (31). Then

$$f = 16P^2 + 9d^2 + 45R^2 + 90W^2 \text{ shall represent } M = 20(9a - b^2).$$

Equations (32) become

$$9x_1 = b + P, \quad 45x_2 = 5b - 4P + 9d,$$
$$108x_3 = 20b - 16P - 9d + 45R,$$
$$180x_4,\ 180x_5 = 20b - 16P - 9d - 45R \pm 90W.$$

Since, $7b^2 \equiv 7P^2 \pmod 9$, by choice of the sign of P, $P \equiv -b \pmod 9$.

Since $P^2 + 4d^2 \equiv 0 \pmod 5$, $P \equiv d \pmod 5$.

From $0 \equiv d^2 + R^2 \pmod 2$, we have $d \equiv R \pmod 2$. Then $0 \equiv 2R^2 + 2W^2 \pmod 4$. Hence $d \equiv R \equiv W \pmod 2$. If d is odd, by choice of sign of R, $d \equiv R \pmod 4$. If d is even, let $d = 2U + R$. Then $0 \equiv 4U(U + R) \pmod 8$. Hence U is even.

The x_i are all integers. They are all $> -k$ if $-20P$, $4(4P - 9d)$, $16P + 9d - 45R$, and $16P + 9d + 45R \pm 90W$ are $< 20(b + 9k)$. The first one and the last two are $\leqq \sqrt{160f}$ by Lemma 13. By comparison with the identity

$$16(4x + 9y)^2 + 16(12x - 3y)^2 \equiv 160(16x^2 + 9y^2)$$

we see that the second $\leqq \sqrt{160f}$. Hence the x_i are all $> -k$ if $\sqrt{160f} < 20(b + 9k)$. This gives (30_2).

If $a \equiv b \pmod 2$ and M is positive, we determine under what conditions $M = 20(9a - b^2)$ is represented by $16P^2 + 9d^2 + 45R^2 + 90W^2$. Put $N = 20(9a - b^2) - 16P^2$. Choose $P \equiv b \pmod 9$ and $\equiv 1 \pmod 5$. Then $N = 9x \equiv 4 \pmod 5$ gives $N = 9(5\ell + 1)$. Jones (7, p. 90) proved that the ternary (1, 5, 10) represents all positive integers $\neq 25^k(5n \pm 2)$. Hence $N/9 = 5\ell + 1$ is represented if positive.

When $M < 16P^2$, N is negative. Since P was chosen modulo 45, then $|P| < 23$. Hence we consider $M < 8464$. Since $-b^2 \equiv 0, 2, 5, 8 \pmod 9$, $9a - b^2$ is of the form $9x + y$ where $y = 0, 2, 5, 8$, and is also even. By computation, we verify that all multiples of 20 of the above form are represented by f except 40. In that case, $9a - b^2 = 2$. In addition, we consider the b's up to 31 inclusive, compute the corresponding a's for which $9a - b^2 = 2$, and find that for these pairs of a and b there are no common

solutions of equations (28); hence the above condition is an actual restriction and not merely a case in which we are not able to make the proof.

Case (1,1,2,2,2). Let $c_1 = c_2 = c_4 = 2$. Cancel 24 from (31). Then

$$f = P^2 + 2d^2 + 2R^2 + 16W^2 \text{ shall represent } M = 3(8a - b^2).$$

Equations (32) become

$$8x_1 = b + P, \quad 24x_2 = 3b - P + 4d, \quad 24x_3 = 3b - P - 2d + 6R,$$
$$24x_4 = 3b - P - 2d - 2R + 8W, \quad 24x_5 = 3b - P - 2d - 2R - 16W.$$

From $P^2 + 2d^2 + 2R^2 + W^2 \equiv 0 \pmod 3$, it follows that $P^2 \equiv d^2$ or $R^2 \pmod 3$. By interchange of d and R if necessary, $P \equiv d \pmod 3$. Then $W \equiv R \pmod 3$.

Since the method fails when b is even, let b be odd. From $P \equiv b \pmod 2$, we have, by choice of the sign of P, $P = 4U - b$. Since $1 \equiv 1 + 2d^2 + 2R^2 \pmod 4$, then $d \equiv R \pmod 2$, and $-3 \equiv 1 + 4R^2 \pmod 8$. Hence $R \equiv 1 \pmod 2$. By choice of the sign of d, $d \equiv R \pmod 4$. Then $8a - 3b^2 \equiv 8Ub + b^2 + 2 + 2 \pmod{16}$. Therefore $2a - b^2 \equiv 2Ub + 1 \pmod 4$, and $U \equiv 0 \pmod 2$.

Hence the x_i are all integers. They are all $> -k$ if $-3P$, $P - 4d$, $P + 2d - 6R$, $P + 2d + 2R - 8W$, $P + 2d + 2R + 16W$ are all $< 3(b + 8k)$. By Lemma 13, the first one and the last two are each $\leqq \sqrt{21f}$. From the two identities,

$$(x + 4y)^2 + 8(x + y)^2 + 2(2x - y)^2 + 4(x - 2y)^2 \equiv 21(x^2 + 2y^2),$$

$$(x + 2y + 6z)^2 + 4(3y - z)^2 + 2(3x - z)^2 + 2(x - y)^2 \equiv 21(x^2 + 2y^2 + 2z^2),$$

we see that $|P| + 4|d|$ and $|P| + 2|d| + 6|R|$ are each $\leqq \sqrt{21f}$. The x_i are all $> -k$ if $\sqrt{21f} < 3(b + 8k)$. This gives (30_2).

It remains to prove that for $a \equiv b \equiv 1 \pmod 2$ and M positive, M is represented by $P^2 + 2d^2 + 2R^2 + 16W^2$. Dickson (6) proved that the ternary (1, 2, 2) represents all positive integers $\neq 4^k(8n + 7)$. Put $W = 0$. Since $M = 3(8a - b^2) \equiv 5 \pmod 8$, it is always represented by f if positive.

Case (1,1,2,2,4). Let $c_1 = 4$, $c_3 = c_4 = 2$. Cancel 30 from (31). Then

$$3(10a - b^2) = 2P^2 + d^2 + 4R^2 + 20W^2.$$

Make the substitution $d = R + 5D$ where $D = x_2 - x_3$. Thus

$$f = 2P^2 + 5T^2 + 20D^2 + 20W^2 \text{ shall represent } M = 3(10a - b^2),$$

where $T = R + D$. Then $R = T - D$ and $d = T + 4D$. Equations (32) become

$$10x_1 = b + P, \quad 30x_2 = 3b - 2P + 5T + 20D,$$
$$30x_3 = 3b - 2P + 5T - 10D, \quad 30x_4 = 3b - 2P - 5T + 10W,$$
$$30x_5 = 3b - 2P - 5T - 20W.$$

Since $b^2 \equiv P^2 \pmod 5$, by choice of the sign of P, $P \equiv -b \pmod 5$.

Note that $P^2 + T^2 + D^2 + W^2 \equiv 0 \pmod 3$. First, let $P^2 \equiv T^2 \equiv D^2 \pmod 3$, $W^2 \equiv 0 \ (3)$, by interchange of D and W if necessary. Choose the signs of T and D so that $T \equiv D \equiv -P \pmod 3$. Second, let $P^2 \equiv D^2 \equiv W^2 \equiv 1 \pmod 3$, $T^2 \equiv 0 \pmod 3$. Choose the signs of D and W so that $P \equiv D \equiv -W \pmod 3$. Third, let $T^2 \equiv W^2 \equiv D^2 \equiv 1 \pmod 3$, $P^2 \equiv 0 \pmod 3$. Choose the signs of D and W so that $T \equiv -D \equiv -W \pmod 3$.

We have $b \equiv T \pmod 2$. Then $2a - 3b^2 \equiv 2P^2 + b^2 \pmod 4$. Hence $a \equiv P^2 \pmod 2$ and $b \equiv P \pmod 2$.

The x_i are therefore all integers. They are all $> -k$ if

$-3P$, $2P - 5T - 20D$, $2P - 5T + 10D$, $2P + 5T - 10W$, and $2P + 5T + 20W$ are $< 3(b + 10k)$. By Lemma 13 they are all $\leqq \sqrt{27f}$. Hence the x_1 are all $> -k$ if $\sqrt{27f} < 3(b + 10k)$. This gives (30_2).

We prove that for $a \equiv b \pmod 2$ and $M = 3(10a - b^2)$ positive, M is represented by $2P^2 + d^2 + 4R^2 + 20W^2$. If b is odd, or if b is even and $3(10a - b^2) \neq 4^k(16n + 14)$ put $W = 0$. Dickson (6) proved that the ternary $(1, 2, 4)$ represents all integers $\neq 4^k(16n + 14)$. In the cases mentioned, $M \neq 4^k(16n + 14)$. If b is even and $3(10a - b^2) = 4^k(16n + 14)$, put $W = 1$. Then

$$3(10a - b^2) = 4^k(16n + 14) - 20 = 4\left\{4^{k-1}(16n + 14) - 5\right\}$$
$$\neq 4^e(16n + 14).$$

It remains to consider $M < 20$, where b is even. The possibilities are $6 = 2\cdot1^2 + 4\cdot1^2$, $12 = 2\cdot2^2 + 4\cdot1^2$ and $18 = 2\cdot3^2$.

Case $(1,1,2,2,5)$. Put $c_1 = 5$, $c_3 = c_4 = 2$. Cancel 3 from (31). Then

$$30(11a - b^2) = 25P^2 + 11d^2 + 44R^2 + 220W^2.$$

Make the substitution $d = R + 5D$ where $D = x_2 - x_3$. Then

$$f = 5P^2 + 11T^2 + 44D^2 + 44W^2 \text{ shall represent } M = 6(11a - b^2)$$

where $T = R + D$. Then $R = T - D$, $d = T + 4D$, and (32) gives

$$11x_1 = b + P, \quad 66x_2 = 6b - 5P + 11T + 44D,$$
$$66x_3 = 6b - 5P + 11T - 22D, \quad 66x_4 = 6b - 5P - 11T + 22W,$$
$$66x_5 = 6b - 5P - 11T - 44W.$$

By choice of the sign of P, $P \equiv -b \pmod{11}$. Also $P^2 \equiv T^2 \pmod 2$. $P^2 + T^2 + D^2 + W^2 \equiv 0 \pmod 3$. First, by interchange of D and W if necessary, take $P^2 \equiv T^2 \equiv D^2$ and $W^2 \equiv 0 \pmod 3$. Choose the signs of T and D so that $T \equiv -P \equiv -D \pmod 3$.

Second, take $P^2 \equiv D^2 \equiv W^2 \equiv 1$ and $T^2 \equiv 0 \pmod 3$. By choice of the signs of D and W, $D \equiv P \equiv -W \pmod 3$. Third, take $T^2 \equiv D^2 \equiv W^2 \equiv 1$ and $P^2 \equiv 0 \pmod 3$. By choice of the signs of D and W, $D \equiv -T \equiv W \pmod 3$.

The x_1 are all integers. They are all $> -k$ if $-6P$, $5P - 11T - 44D$, $5P - 11T + 22D$, $5P + 11T - 22W$, and $5P + 11T + 44W$ are $< 6(b + 11k)$. By Lemma 13, all of these are $\leqq \sqrt{60f}$. Hence the x_1 are all $> -k$ if $\sqrt{60f} < 6(b + 11k)$. This gives (30_2).

We determine under what conditions with $a \equiv b \pmod 2$ and $M = 6(11a - b^2)$ positive, M is represented by $5P^2 + 11T^2 + 44D^2 + 44W^2$. Put $N = M - 5P^2$. Choose $P \equiv b \pmod{11}$ and $\equiv 1 \pmod 2$. Then $N \equiv 0 \pmod{11}$ and $\equiv -1 \pmod 4$. Hence $N = 44\ell + 11$. Jones (7, p. 75) proved that the ternary (1, 4, 4) represents all positive integers $\neq 4n + 2$, $4n + 3$, $4^k(8n + 7)$. Since $N/11 = 4\ell + 1$ is not one of these types it is represented by (1, 4, 4) if positive.

We choose P modulo 22 and hence $|P| < 11$. Thus N is negative if $M < 605$. Since $-b^2 \equiv 0, 2, 6, 7, 8, 10 \pmod{11}$, $11a - b^2 = 11a + y$, $(y = 0, 2, 6, 7, 8, 10)$. Also $11a - b^2$ is even. By actual computation, we verify that the only values of M which are not represented are 12, 36, 48, which make $11a - b^2 = 2, 6, 8$. In order to determine that these are actual exceptions, we consider all values of b up to 30 inclusive, determine the a's for which $11a - b^2 = 2, 6, 8$ and find that for the respective pairs there are no common solutions of equations (28).

Case (1,1,2,4,4). Let $c_1 = c_3 = 4$, $c_2 = 2$. Cancel 48 from (31), obtaining

$$2(12a - b^2) = P^2 + d^2 + 8R^2 + 12W^2.$$

Put $P = d + 8B$ where $B = x_1 - x_2$. Then

$f = T^2 + 4R^2 + 6W^2 + 16B^2$ shall represent $M = 12a - b^2$

where $T = d + 4B$. Then $d = T - 4B$, $P = T + 4B$. Equations (32) become

$$12x_1 = b + T + 4B, \quad 12x_2 = b + T - 8B, \quad 12x_3 = b - T + 2R,$$
$$12x_4, \quad 12x_5 = b - T - 4R \pm 6W.$$

Evidently $2b^2 \equiv T^2 + R^2 + B^2 \pmod 3$. First, let $b \equiv 0 \pmod 3$. Then $T^2 \equiv B^2 \equiv R^2 \pmod 3$. By choice of the signs of R and B, $T \equiv -B \equiv -R \pmod 3$. Second, let $b^2 \equiv 1 \pmod 3$. If $T^2 \equiv R^2 \equiv 1$ and $B^2 \equiv 0 \pmod 3$, by choice of the signs of T and R, $T \equiv R \equiv -b \pmod 3$. If $T^2 \equiv B^2 \equiv 1$ and $R^2 \equiv 0 \pmod 3$, by choice of the signs of T and B, $T \equiv B \equiv b \pmod 3$. If $R^2 \equiv B^2 \equiv 1$ and $T^2 \equiv 0 \pmod 3$, by choice of the signs of B and R, $B \equiv -R \equiv -b \pmod 3$.

Since the method fails when b is even, let b be odd. Then $b \equiv T \equiv W \equiv 1 \pmod 2$. By choice of the sign of T, $T \equiv -b \pmod 4$. Since $4 - 1 \equiv 1 + 4R^2 + 6 \pmod 8$, $R \equiv b \pmod 2$.

Hence the x_i are all integers. They are $> -k$ if $-T - 4B$, $-T + 8B$, $T - 2R$, $T + 4R \pm 6W$ are all $< b + 12k$. By Lemma 13, these quantities are all $\leqq \sqrt{11f}$. Hence the x_i are all $> -k$ if $\sqrt{11f} < b + 12K$. This gives (30_2).

We shall prove that for a, b odd, $M = 12a - b^2$ positive and $\neq 3$, M is represented by f. Jones (7, p. 76) proved that the ternary (1, 4, 6) represents all positive integers $\neq 16n + 2$, $9^k(9n + 3)$. Since M is odd, $M \neq 16n + 2$. If $b \not\equiv 0 \pmod 3$, $M \neq 9^k(9n + 3)$. Hence let $b \equiv 0 \pmod 3$. Choose $B \equiv \pm 1 \pmod 3$. Then $N = 12(a - b^2) - 16B^2 \not\equiv 0 \pmod 3$ is odd, and hence is represented by (1, 4, 6) if positive. N is negative if $M < 16B^2 < 16$, since B was chosen $\equiv \pm 1 \pmod 3$. The possible values of M

are 3, which is not represented, $9 = 3^2$, and $15 = 3^2 + 6\cdot 1^2$. We consider, in addition, the b's up to 33 inclusive, determine the corresponding a's for which $12a - b^2 = 3$, and find that for the respective sets of a and b there are no common solutions x_1. This completes the proof of Theorem 2.

13. Proof of Theorem 3. We shall use five Cauchy lemmas, of which Dickson (4) proved the first four and Chatland (8) the fifth:

LEMMA 14. *Let $(c_1,\ldots,c_4)$ be one of the four sets below. Let (28) hold and*

$$ta \geqq b^2, \qquad (t-1)a < b^2 + 2bk + tk^2.$$

Then there exist integral solutions $> -k$ of (28) with $s = 4$ if the following further conditions hold:

$(1,1,1,4)$, $7a - b^2 \neq 4^m(8n+7)$.

$(1,1,1,5)$, b odd, $3(8a - b^2) \neq 25^m(25n \pm 10)$.

$(1,1,2,4)$, b odd or double an odd.

$(1,2,2,2)$, $7a - b^2 \neq 4^m(16n+14)$.

$(1,2,2,4)$, $9a - b^2 \neq 4^k(8n+7)$.

The method of proof for Theorem 3 is the same as a proof used by Dickson (4, p. 526).

Theorem 3 follows from Theorem 2 unless

$$(33) \qquad (t-1)a \geqq b^2 + 2b + t, \qquad (t-2)(a-2) < b^2.$$

First, put $c_5 = 1$, $x_5 = 1$. Let $\alpha = a - 1 = \sum_{i=1}^{4} c_i x_i^2$, $\beta = b - 1 = \sum_{i=1}^{4} c_i x_i$. Then $\alpha \equiv \beta \pmod 2$. Also $\beta \geqq 0$ unless $b = 0$. Then (33) gives $2 \leqq a < 2$, which is impossible. From (33_1), we obtain $(t-1)\alpha \geqq \beta^2 + 4\beta + 2$ and thus $\geqq \beta^2$, and from (33_2),

$(t - 2)\alpha < \beta^2 + 2\beta + t - 1$. Thus α and β satisfy the conditions of Lemma 14. Second, put $c_5 = 1$, $x_5 = 0$. Then $a \equiv b \pmod 2$, $b \geqq 0$. By (33_1), $(t - 1)a \geqq b^2$. By (33_2), $(t - 2)a < b^2 + 2t - 4 \leqq b^2 + 2b + t - 1$ if $t - 3 \leqq 2b$. Since $t = 8, 9, 10$, this is true unless $b = 0, 1, 2, 3$. The case $b = 0$ has already been discussed. When $b = 1$, $2 \leqq a \leqq 2$; but $a \equiv b \pmod 2$. When $b = 2$, $2 < a \leqq 2$ for $t = 8, 9$; $2 \leqq a \leqq 2$ for $t = 10$. When $a = b = 2$, Theorem 3 holds. When $b = 3$, $3 < a \leqq 3$ for $t = 8$; $3 \leqq a \leqq 3$ for $t = 9$; $3 \leqq a \leqq 3$ for $t = 10$. For $a = b = 3$, Theorem 3 holds. Thus, except for $b = 0, \ldots, 3$, a and b satisfy the conditions of Lemma 14.

Case (1,1,1,1,4). If b is odd and $7a - b^2 \neq 4^m(8n + 7)$, put $c_5 = 1$, $x_5 = 0$. If b is odd and $7a - b^2 = 4^m(8n + 7)$, put $c_5 = 1$, $x_5 = 1$. Then $7\alpha - \beta^2 = 4^m(8n + 7) + 2b - 8$. If $m = 0, 1$, $m \geqq 2$, we have $7\alpha - \beta^2 = 8\ell + 2b + 7$, $8\ell' + 2b + 4$, $8\ell'' + 2b$ respectively, and no one of them is of the form $4^k(8\ell + 7)$.

Case (1,1,1,1,5). If b is odd and $3(8a - b^2) \neq 25^m(25n \pm 10)$, put $c_5 = 1$, $x_5 = 0$. If b is even, put $c_5 = 1$, $x_5 = 1$; then the conditions of Lemma 14 are satisfied unless $3\{8(a - 1) - (b - 1)^2\} = 25^m(25n \pm 10)$.

Case (1,1,1,2,4). If b is odd, put $c_5 = 1$, $x_5 = 0$. If b is even, put $c_5 = 1$, $x_5 = 1$, whence β is odd.

Case (1,1,2,2,2). If b is odd and $7a - b^2 \neq 4^m(16n + 14)$, put $c_5 = 1$, $x_5 = 0$.

Case (1,1,2,2,4). If $9a - b^2 \neq 4^k(8n + 7)$, put $c_5 = 1$, $x_5 = 0$. If b is even and $9a - b^2 = 4^k(8n + 7)$, put $c_5 = 1$, $x_5 = 1$. Then $9\alpha - \beta^2 = 4^k(8n + 7) + 2b - 10$. If $k = 0$, $9\alpha - \beta^2$ becomes $8n + 7 + 2b - 10$. If b is even, this is not of the form $8\ell + 7$ since that would imply $b \equiv 1 \pmod 8$. If $k = 1$, $9\alpha - \beta^2$

$= 8\ell + 2(b + 1)$ with b even. If $k \geqq 2$, $9\alpha - \beta^2 = 8\ell + 2(b - 1)$, with b even. This concludes the proof of Theorem 3.

14. Proof of Theorems 4, 5, and 6. The method used in the proof of Theorem 4 is the same as that used for Theorem 3, except that here we use the results of Theorem 2. First, put $c_6 = 1$, $x_6 = 1$. Let $\alpha = a - 1$, $\beta = b - 1$. Then $\alpha \equiv \beta \pmod 2$, and $\beta \geqq 0$ except when $b = 0$. But, since $ta \geqq e$, $(t - 2)(a - 2) < 0$, and $a \equiv b \pmod 2$, $b = 0$ gives no a. From $(t - 1)(\alpha + 1) \geqq \beta^2 + 2\beta + 1 + e$, we have $(t - 1)\alpha \geqq \beta^2$ if $2\beta \geqq t - 2 - e$. Since $(t - 2)(\alpha - 1) < \beta^2 + 2\beta + 1$, then $(t - 2)\alpha < \beta^2 + 2\beta + t - 1$. Second, put $c_6 = 1$, $x_6 = 0$. Then $(t - 1)a \geqq b^2 + e > b^2$; and $(t - 2)a < b^2 + 2t - 4 \leqq b^2 + 2b + t - 1$ if $2b \geqq t - 3$.

Case (1,1,1,1,1,4). When b is even, put $c_6 = 1$, $x_6 = 1$, whence β is odd. We note that $2\beta \geqq t - 2 - e = 6$ except when $b = \beta + 1$ is 1, 2, 3. Since $7(a - 2) < b^2$ and $8a \geqq b^2 + 1$, $b = 1$ gives $a = 1$; $b = 2$, $a = 2$; $b = 3$, $a = 3$. In all of these cases Theorem 4 holds. When b is odd, put $c_6 = 1$, $x_6 = 0$. Then $2b \geqq t - 3 = 6$ except when $b = 0, 1, 2$, cases which have already been discussed.

Case (1,1,1,1,1,5). Put $c_6 = 1$, $x_6 = 0$. Then $2b \geqq 7$ except when $b = 0, \ldots, 3$. When $b = 0$, there is no a; when $b = 1$, $a = 1$; when $b = 2$, $a = 2$; when $b = 3$, $a = 3$.

Case (1,1,1,2,2,2). When b is even, put $c_5 = 1$, $x_5 = 1$. Then $2\beta \geqq 6$ except when $b = 1, 2, 3$. When $b = 1$, then $a = 1$; when $b = 2$, $a = 2$; when $b = 3$, $a = 3$. When b is odd, put $c_5 = 1$, $x_5 = 0$. Then $2b \geqq 6$ except when $b = 0, 1, 2$, cases already discussed.

Case (1,1,1,2,2,4). Put $c_6 = 1$, $x_6 = 0$. $2b \geqq 8$ except when $b = 0, \ldots, 3$. When $b = 0, 3$, there is no a; when $b = 1$, $a = 1$; when $b = 2$, $a = 2$.

Case (1,1,1,1,2,4). Put $c_6 = 1$, $x_6 = 0$. Then $2b \geqq 7$ except when $b = 0, \ldots, 3$. When $b = 0$, there is no a; when $b = 1$, $a = 1$; when $b = 2$, $a = 2$; when $b = 3$, $a = 3$. For this case $e = 1$ was sufficient, but $9a - b^2 \neq 1$ and by hypothesis $9a - b^2 \neq 2$. Hence we made $e = 3$.

Case (1,1,1,2,2,5). Put $c_6 = 1$, $x_6 = 0$. Then $2b \geqq 9$ except when $b = 0, \ldots, 4$. When $b = 0, 3, 4$, there is no a; when $b = 1$, $a = 1$; when $b = 2$, $a = 2$. For this case $e = 7$ was sufficient, but, since $11a - b^2 \neq 8$, we took $e = 9$ in the theorem.

Case (1,1,1,2,4,4). When b is even, put $c_6 = 1$, $x_6 = 1$. Then $2\beta \geqq 2$ except when $b = 1, 2$. When $b = 1$, $a = 1$; when $b = 2$, $a = 2$. When b is odd, put $c_6 = 1$, $x_6 = 0$. Then $2b \geqq t - 3$ except when $b = 0, \ldots, 4$. The cases $b = 0, 1, 2$ have already been discussed. When $b = 3, 4$, there is no a.

The proof for Theorems 5 and 6 is exactly the same as a proof given by Dickson (4, p. 528) except that, for Theorem 6, we use $(t - 1)a \geqq b^2 + e$ instead of $ta \geqq b^2$.

PART IV

DETERMINATION OF UNIVERSAL FUNCTIONS BY USE OF FIVE-TERM CAUCHY LEMMAS

15. Introduction. Although the limits in Theorem 5 are more advantageous for our use than those in Theorem 2, we cannot apply all of Theorem 5 because of the conditions imposed on a and b. So in this section we shall apply three of the cases in Theorem 2, four of those in Theorem 5, and four others which were proved by Dickson (4). As a result we shall lower the limit for A in certain cases, and in four of these cases we shall obtain universal theorems. However, we shall be forced at times to impose a stronger condition on a_h than was necessary for Theorem 1.

Dickson (4) proved the following result:

LEMMA 15. For (1,1,1,1,j), j = 1, 2, 3, and (1,1,1,2,2), equations (28) with s = 5 have integral solutions > -2 if we assume

$$a,\ b \text{ integers}, \quad b \geqq -t, \quad a \equiv b \pmod 2,$$
$$ta \geqq b^2, \quad (t-2)(a-2) < b^2 + 4b + 2t.$$

As in Section 3, we replace the limits on a and b by a limit on A involving m and obtain

$$U = 8m\{(t-2)A - 2m(t-1)\} + h^2, \quad h = 4m + n(t-2),$$
$$V = 8mt(A-E) + n^2t^2, \quad P = 2md - 4m + 2n, \tag{34}$$
$$F \equiv (U + V - P^2)^2 - 4UV \geqq 0.$$

If $n \geqq 0$, the minor conditions are

$$2A \geqq tE, \qquad (t - 2)A \geqq 2m(t - 1).$$

If $n < 0$, we have in addition

$$8m(t - 2)A \geqq 16m^2(t - 1) + nt(4n - nt - 8m).$$

We shall need

LEMMA 16. *Let* $f = (1,1,1,1,a_5,\ldots,a_n)$, $a_5 = 1, 2, 3, 5$, *or* $(1,1,1,2,a_5,\ldots,a_n)$, $a_5 = 2, 4$, *or* $(1,1,2,2,a_5,\ldots,a_n)$, $a_5 = 4, 5$, *where* $a_k \leqq w_{k-1} + 1$ $(6 \leqq k \leqq n)$. *Also let* β *be any integer*, $0 \leqq \beta \leqq A$. *Then there exist integers*, a, b, r, *each* $\geqq 0$, *such that* $A = m(a - b)/2 + b + r$, $a \equiv b \pmod 2$, F_6 *represents* $r \leqq m - 2 - t$, *and* b *is one of* $\beta, \ldots, \beta + t + 1$.

Write $A = mg + \beta + \rho$. If F_6 represents ρ, put $r = \rho$, $b = \beta$, $a = 2g + b$. Now $a_k \leqq w_{k-1} + 1 = w_{k-1}' + t + 1$. Hence if F_6 does not represent ρ where $0 < \rho < m - 2 - t$, Lemma 6 states that F_6 represents $\rho - s$, where $s = 1, \ldots,$ or t. Hence put $r = \rho - s$, $b = \beta + s$, $a = 2g + b$. If $\rho = m - s$ $(s = 1, \ldots, t + 1)$, put $r = \rho - (2 + t - s)$, $b = \beta + (2 + t - s)$, $a = 2g + b$, since F_6 represents $\rho - (2 + t - s)$.

Unless otherwise stated we use Lemma 15 or Theorem 5, both of which give rise to (34).

16. *Functions Universal for All Integers.* First, let $f = (1,1,1,1,1,a_6,\ldots,a_n)$. Here $d = 7$, $t = 5$, $E = m - 7$. Hence

$$U = 3(8mA - 21m^2 + 4m + 12), \quad V = 5(8mA - 3m^2 + 36m + 20),$$

$$P = 4(2m + 1), \quad U + V - P^2 = 2(32mA - 71m^2 + 64m + 60),$$

$$F = 256(m^2A^2 - 26m^3A - 11m^2A + 64m^4 + 38m^3 + 4m^2).$$

Then $F \geqq 0$ if $A \geqq 26m + 11$, a value which also satisfies the minor conditions. We then examine

TABLE I

SUMS BY FIVE OF p(x)

0 to 5, m - 1 to m + 6, 2m - 2 to 2m + 7, 3m - 3 to 3m + 8, 4m - 4, 4m - 3, 4m - 1, 4m, 4m + 2 to 9, 5m - 5, 5m - 2, 5m + 1 to 10, 6m, 6m + 1, 3 to 10, 7m - 1, 7m + 2 to 11, 8m + 2 to 11, 9m + 1 to 12, 10m, 10m + 3, 5 to 12, 11m + 4 to 13, 12m + 3 to 13, 13m + 2, 3, 5 to 14, 14m + 1, 4, 6 to 14, 15m + 5 to 15, 16m + 5 to 15, 17m + 4 to 15, 18m + 3, 4, 6 to 16, 19m + 2, 5, 6, 8 to 16, 20m + 7, 8, 10 to 16, 21m + 6 to 17, 22m + 6 to 17, 23m + 5 to 17, 24m + 4, 5, 7, 8, 10 to 18, 25m + 3, 6, 9 to 18, 26m + 8 to 18, 27m + 7, 9 to 19.

Since we have already proved that f represents A where $0 \leqq A \leqq 15m + 6$, we consider only the interval from 15m + 6 to 26m + 11. Think of the table as being divided into blocks, all numbers in a block displaying the same multiple of m. The missing numbers in a block all exceed some number in the preceding block by m - 7. Consider any number s in the interval between two consecutive blocks. At the end of every block and hence at the end of the first one above there are six consecutive integers. Denote the smallest of these by h. If F_5 does not represent s - h where $s - h < m - 7$, it represents one of s - (h + 1), ..., s - (h + 5). Since the difference between the first number of one block and the last number of the preceding block $\leqq m - 9$, then if $s - h > m - 7$, one of s - (h + 1), ..., s - (h + 5) equals m - 7 and hence is represented by F_6. This proves the first part of Theorem 7.

Next let $f = (1,1,1,1,2,a_6,\ldots,a_n)$. Hence t = 6, d = 8, E = m - 8, and

$$U = 16(2mA - 5m^2 + 4), \qquad V = 12(4mA - m^2 + 20m + 12),$$
$$P = 2(2 + 5m), \qquad U + V - P^2 = 16(5mA - 12m^2 + 10m + 12),$$
$$F = 256(m^2A^2 - 54m^3A - 20m^2A + 129m^4 + 60m^3 + 4m^2).$$

The minor conditions hold and $F \geqq 0$ if $A \geqq 54m + 20$. We now examine

TABLE II

VALUES OF $p_1 + p_2 + p_3 + p_4 + 2p_5$

0 to 6, m - 1 to m + 7, 2m - 2 to 2m + 8, 3m - 3 to 3m + 9, 4m - 4 to 4m + 10, 5m - 5, 5m - 4, 5m - 2, 5m - 1, 5m + 1 to 11, 6m - 6, 6m - 3, 6m to 6m + 12, 7m - 1, 7m, 7m + 2 to 12, 8m - 2, 8m + 1 to 13, 9m + 1 to 13, 10m to 10m + 14, 11m - 1, 11m + 2, 4 to 14, 12m + 3 to 15, 13m + 2 to 15, 14m + 1, 2, 4, to 16, 15m, 15m + 3, 5 to 16, 16m + 4 to 17, 17m + 4 to 17, 18m + 3 to 18, 19m + 2, 3, 5 to 18, 20m + 1, 4, 5, 7 to 18, 21m + 6 to 19, 22m + 5 to 19, 23m + 5 to 19, 24m + 4 to 20, 25m + 3, 4, 6, 7, 9 to 20, 26m + 2, 5, 8 to 20, 27m + 7 to 21, 28m + 6, 8 to 21, 29m + 7 to 21, 30m + 6 to 22, 31m + 5 to 22, 32m + 4, 5, 7, 8, 10 to 22, 33m + 3, 6, 9 to 23, 34m + 8 to 23, 35m + 7, 8, 10 to 23, 36m + 9 to 24, 37m + 8 to 24, 38m + 7 to 24, 39m + 6 to 24, 40m + 5, 6, 8, 9, 11 to 25, 41m + 4, 7, 10 to 25, 42m + 9, 10, 12 to 25, 43m + 8, 11 to 25, 44m + 11 to 26, 45m + 10 to 26, 46m + 9 to 26, 47m + 8 to 26, 48m + 7 to 27, 49m + 6, 7, 9 to 27, 50m + 5, 8, 10 to 27, 51m + 10, 11, 13 to 27, 52m + 9, 12 to 28, 53m + 11 to 28, 54m + 11 to 28.

We check for $15m + 6 < A \leqq 54m + 20$. Every number missing within a block is the sum of some number in the preceding block and m - 8. Let s be a number between two blocks. Denote

the smallest of the seven consecutive integers at the end of the first block by h. Then if F_6 does not represent $s - h < m - 8$, it represents one of $s - (h + 1), \ldots, s - (h + 6)$. Since the difference between the first number of one block and the last number of the preceding block $\leqq m - 12$, then, if $s - h > m - 8$, one of $s - (h + 1), \ldots, s - (h + 6)$ equals $m - 8$ and hence is represented.

If $f = (1,1,1,1,2)$, then $m = w + 2 = 8$, $r = 0$. Lemma 16 and the computation still hold. Hence $F \geqq$ if $A \geqq 452$. Examination of Table II with $m = 8$ reveals that it includes all integers A where $0 < A < 452$.

Since $t = 7$ for both $f = (1,1,1,1,3,a_6,\ldots,a_n)$ and $f = (1,1,1,2,2,a_6,\ldots,a_n)$, we study these cases together. Since $d = 9$, $t = 7$, $E = m - 9$, (34) gives

$$U = 5(8mA - 19m^2 - 4m + 20), \quad V = 7(8mA - m^2 + 44m + 28),$$
$$P = 4(3m + 1), \quad U + V - P^2 = 2(48mA - 123m^2 + 96m + 140),$$
$$F = 256(m^2A^2 - 97m^3A - 31m^2A + 226m^4 + 86m^3 + 4m^2).$$

Hence $F \geqq 0$ and the minor conditions hold if $A \geqq 97m + 31$. We then examine Tables III and IV.

TABLE III

VALUES OF $p_1 + p_2 + p_3 + p_4 + 3p_5$

0 to 7, m - 1 to m + 8, 2m - 2 to 2m + 9, 3m - 3 to 3m + 1, 3 to 10, 4m - 4 to 4m + 11, 5m - 5 to 5m + 12, 6m - 6, -5, -3, -2, 6m to 6m + 13, 7m - 7, -4, -1, 7m to 7m + 14, 8m - 2, -1, 8m + 1 to 14, 9m - 3, 9m to 9m + 15, 10m to 10m + 15, 11m - 1 to 11m + 16, 12m - 2, 12m + 1, 3 to 16, 13m + 2 to 17, 14m + 1 to 17, 15m, 15m + 1, 3 to 18, 16m - 1, 16m + 2, 4 to 18, 17m + 3 to 19, 18m + 3 to 19,

TABLE III--CONTINUED

19m + 2 to 20, 20m + 1, 2, 4 to 20, 21m, 21m + 3, 4, 6 to 21, 22m + 5 to 21, 23m + 4 to 21, 24m + 4 to 22, 25m + 3 to 22, 26m + 2, 3, 5, 6, 8 to 22, 27m + 1, 4, 7 to 23, 28m + 6 to 23, 29m + 5, 7 to 23, 30m + 6 to 24, 31m + 5 to 24, 32m + 4 to 24, 33m + 3, 4, 6, 7, 9 to 25, 34m + 2, 5, 8 to 25, 35m + 7 to 25, 36m + 6, 7, 9 to 26, 37m + 8 to 26, 38m + 7 to 26, 39m + 6 to 27, 40m + 5 to 27, 41m + 4, 5, 7, 8, 10 to 27, 42m + 3, 6, 9 to 28, 43m + 8, 9, 11 to 28, 44m + 7, 10 to 28, 45m + 10 to 28, 46m + 9 to 29, 47m + 8 to 29, 48m + 7 to 10, 12 to 29, 49m + 6 to 29, 50m + 5, 6, 8 to 30, 51m + 4, 7, 9 to 30, 52m + 9, 10, 12 to 30, 53m + 8, 11 to 30, 54m + 10 to 14, 16 to 31, 55m + 10 to 31, 56m + 9 to 31, 57m + 9, 10, 12 to 31, 58m + 8 to 32, 59m + 7 to 32, 60m + 6, 7, 9, 10, 12 to 32, 61m + 5, 8, 11 to 32, 62m + 10, 11, 13 to 33, 63m + 9, 12 to 33, 64m + 12 to 33, 65m + 11 to 33, 66m + 10, 12 to 34, 67m + 11 to 34, 68m + 10, 11, 13 to 34, 69m + 9 to 34, 70m + 8 to 35, 71m + 7, 8, 10 to 35, 72m + 6, 9, 12 to 35, 73m + 11, 12, 14 to 35, 74m + 10, 13, 14, 16 to 35, 75m + 12 to 36, 76m + 12 to 36, 77m + 11, 13 to 36, 78m + 13 to 36, 79m + 12 to 36, 80m + 11 to 37, 81m + 10 to 13, 15 to 37, 82m + 9 to 37, 83m + 8, 9, 11, 12, 14 to 37, 84m + 7, 10, 13 to 37, 85m + 12 to 38, 86m + 11, 13 to 38, 87m + 14 to 38, 88m + 13 to 38, 89m + 12, 15, 17 to 38, 90m + 16 to 39, 91m + 14 to 39, 92m + 13 to 39, 93m + 12, 13, 15 to 39, 94m + 11 to 14, 16 to 39, 95m + 10 to 40, 96m + 9, 10, 12, 13, 15 to 40, 97m + 8, 11, 14 to 40, 98m + 13, 14, 16 to 40, 99m + 12, 15, 16, 18 to 40, 100m + 15 to 41.

TABLE IV

VALUES OF $p_1 + p_2 + p_3 + 2p_4 + 2p_5$

0 to 7, m - 1 to m + 8, 2m - 2 to 2m + 9, 3m - 3 to 3m + 10, 4m - 4 to 4m + 11, 5m - 5 to 5m + 12, 6m - 6, -5, -3, -2, 6m to 6m + 13, 7m - 7, -4, -1, 7m to 7m + 14, 8m - 2, -1, 8m + 1 to 14, 9m - 3, 9m to 9m + 15, 10m to 10m + 3, 5 to 15, 11m - 1 to 11m + 16, 12m - 2, 12m + 1, 3 to 16, 13m + 2 to 17, 14m + 1 to 17, 15m, 15m + 1, 3 to 18, 16m - 1, 16m + 2, 4 to 18, 17m + 3 to 19, 18m + 3 to 19, 19m + 2 to 20, 20m + 1, 2, 4 to 20, 21m, 21m + 3, 4, 6 to 21, 22m + 5 to 21, 23m + 4 to 21, 24m + 4 to 22, 25m + 3 to 22, 26m + 2, 3, 5, 6, 8 to 22, 27m + 1, 4, 7, 9 to 23, 28m + 6, 7, 8 to 23, 29m + 5, 7 to 23, 30m + 6 to 24, 31m + 5 to 24, 32m + 4 to 7, 9 to 24, 33m + 3, 4, 6, 7, 9 to 25, 34m + 2, 5, 8 to 25, 35m + 7 to 11, 13 to 25, 36m + 6, 7, 9 to 26, 37m + 8 to 26, 38m + 7 to 26, 39m + 6 to 27, 40m + 5 to 27, 41m + 4, 5, 7, 8, 10 to 27, 42m + 3, 6, 9 to 28, 43m + 8, 9, 11 to 28, 44m + 7, 10 to 28, 45m + 10 to 28, 46m + 9 to 29, 47m + 8 to 29, 48m + 7 to 29, 49m + 6 to 9, 11 to 29, 50m + 5, 6, 8 to 30, 51m + 4, 7, 9, 10, 12 to 30, 52m + 9, 10, 12 to 30, 53m + 8, 11 to 30, 54m + 10 to 31, 55m + 10 to 31, 56m + 9 to 31, 57m + 9 to 31, 58m + 8 to 32, 59m + 7 to 32, 60m + 6, 7, 9, 10, 12 to 32, 61m + 5, 8, 11 to 32, 62m + 10, 11, 14 to 33, 63m + 9, 12 to 33, 64m + 12 to 15, 17 to 33, 65m + 11 to 33, 66m + 10, 12 to 34, 67m + 11 to 34, 68m + 10 to 34, 69m + 9 to 34, 70m + 8 to 11, 13 to 35, 71m + 7, 8, 10 to 35, 72m + 6, 9, 12, 14 to 35, 73m + 11, 12, 14 to 35, 74m + 10, 13 to 35, 75m + 12, 13, 15 to 36, 76m + 12 to 15, 17 to 36,

TABLE IV--CONTINUED

77m + 11, 13 to 36, 78m + 13 to 36, 79m + 12 to 36, 80m + 11 to 37, 81m + 10 to 37, 82m + 9 to 12, 14 to 37, 83m + 8, 9, 11, 12, 14 to 37, 84m + 7, 10, 13, 15 to 37, 85m + 12 to 38, 86m + 11, 13 to 38, 87m + 14 to 17, 19 to 38, 88m + 13 to 15, 17 to 38, 89m + 12, 15, 17 to 38, 90m + 16 to 39, 91m + 14 to 39, 92m + 13 to 39, 93m + 12 to 39, 94m + 11 to 39, 95m + 10 to 13, 15 to 40, 96m + 9, 10, 12, 13, 15 to 40, 97m + 8, 11, 14 to 40, 98m + 13, 14, 16 to 40, 99m + 12, 15 to 40, 100m + 15, 17 to 41.

Consider the integers A where $15m + 6 < A < 97m + 31$. The missing numbers in any block differ from some number in the preceding block by $m - 9$. Let s be a number between two consecutive blocks. Denote the smallest of the eight consecutive integers at the end of the first block by h. If F_6 does not represent $s - h < m - 9$, it represents one of $s - (h + 1), \ldots, s - (h + 7)$. Since the difference between the first number of one block and the last number of the preceding block $\leqq m - 15$, then, if $s - h > m - 9$, one of $s - (h + 1), \ldots, s - (h + 7) = m - 9$, which is represented by F_6.

Since (1,1,1,1,3) does not represent $3m + 2$, it is not universal. We shall prove f = (1,1,1,2,2) is universal. Here $m = 9$, $r = 0$. Lemma 16 holds and hence f is universal except for $141 < A < 904$. Because the missing numbers in Table IV differ from some number in the preceding block by $m - 9$, it follows that they lie in the preceding block. Since the first number in one block exceeds the last number in the preceding block by a quantity $\leqq m - 15$, the blocks must overlap.

17. Determination of Lower Limits for Certain Universal

Functions. For $f = (1,1,1,1,5,a_6,\ldots,a_n)$, apply Theorem 2. Then $t = 9$, $d = 11$, $E = m - 11$ give

$$U = 64(mA - 2m^2 - 2m + 4), \quad V = 9(8mA + m^2 + 52m + 36),$$
$$P = 2 + 17m, \quad U + V - P^2 = 8(17mA - 51m^2 + 34m + 72),$$
$$F = 64(m^2A^2 - 1194m^3A - 140m^2A + 2673m^4 + 348m^3 + 4m^2).$$

The minor conditions hold and $F \geqq 0$ if $A \geqq 1194m + 140$, thus decreasing the lower limit in Theorem 1 by $70m + 20$.

Let $f = (1,1,1,2,4,a_6,\ldots,a_n)$. Theorem 5 states that $9a - b^2 \neq 2$; hence use instead of the inequality $ta \geqq b^2$, $ta \geqq b^2 + 3$, which implies the former. This involves changing the value of V in (34) and also changing one of the minor conditions. Then

$$V = 8mt(A - E) + n^2t^2 - 12m^2, \quad 4A \geqq 2tE + 3m.$$

Since $d = 11$, $t = 9$, $E = m - 11$, we have

$$U = 56mA - 119m^2 - 84m + 96, \quad V = 3(24mA - m^2 + 156m + 108),$$
$$P = 4(4m + 1), \quad U + V - P^2 = 2(64mA - 189m^2 + 128m + 252),$$
$$F = 16(16m^2A^2 - 3864m^3A - 944m^2A + 8841m^4 + 1764m^3 - 104m^2).$$

Hence $F \geqq 0$ and the minor conditions are satisfied if $A \geqq 241\frac{1}{2}m + 59$. In Theorem 1, $A \geqq 1390m + 180$.

If $f = (1,1,1,2,4)$, Lemma 16 and the computation above hold. Since $m = 11$, $(1,1,1,2,4)$ is universal for $A \geqq 2716$.

For $f = (1,1,2,2,4,a_6,\ldots,a_n)$ use Theorem 2. Substituting $t = 10$, $d = 12$, $E = m - 12$ in (11), (13), (18), (19) gives

$$U = 9(8mA - 15m^2 - 20m + 36), \quad V = 20(4mA + m^2 + 28m + 20),$$
$$P = 19m + 2, \quad U + V - P^2 = 4(38mA - 119m^2 + 76m + 180),$$
$$F = 64(m^2A^2 - 1676m^3A - 176m^2A + 3709m^4 + 428m^3 + 4m^2).$$

The minor conditions hold and $F \geqq 0$ if $A \geqq 1676m + 176$. Theorem 1 requires $A \geqq 1984m - 18$.

In order to remove the restriction $11a - b^2 \neq 2, 6, 8$ on $f = (1,1,2,2,5,a_6,\ldots,a_n)$ in Theorem 2, we replace (30_1) by $ta \geqq b^2 + 9$. This changes (13) and (20) to

$$V = 8mt(A - E) + n^2t^2 - 36m^2, \qquad 2A \geqq 2tE + 9m.$$

Then $d = 13$, $t = 11$, $E = m - 13$ give

$$U = 20(4mA - 7m^2 + 12m + 20), \qquad V = 88mA - 3m^2 + 660m + 484,$$
$$P = 2 + 21m, \qquad U + V - P^2 = 8(21mA - 73m^2 + 42m + 110),$$
$$F = 16(4m^2A^2 - 9124m^3A - 864m^2A + 21211m^4 - 1608m^3 - 344m^2).$$

Hence $F \geqq 0$ and the minor conditions hold if $A \geqq 2281m + 216$, thus decreasing the result in Theorem 1 by $341m - 190$.

If $f = (1,1,2,2,5)$ the above holds and f is universal for $A \geqq 29869$.

Since the method fails for $f = (1,1,1,1,4,a_6,\ldots,a_n)$, $(1,1,2,2,2,a_6,\ldots,a_n)$, and $(1,1,2,2,4,a_6,\ldots,a_n)$ when $a_6, \ldots, a_n$ are all even, we assume for these values of f that

(35) among $a_6, \ldots, a_n$ there is a first coefficient a_h which is odd.

Here we need

LEMMA 17. *Let* $f = (1,1,1,1,4,a_6,\ldots,a_n)$ *or* $(1,1,2,2,2,a_6,\ldots,a_n)$, *where* $a_k \leqq w_{k-1} + 1$ $(6 \leqq k \leqq n)$, *satisfy* (35) *and let* β *be any odd integer,* $a_h - 1 \leqq \beta \leqq A$. *Then there exist integers* a, b, r, *each* $\geqq 0$, *such that* $A = m(a - b)/2 + b + r$, $a \equiv b \pmod 2$, $b \equiv 1 \pmod 2$, F_6 *represents* $r \leqq m - 2 - t$, *and* b *is one of* $\beta - a_h + 1, \ldots, \beta + a_h + 1 + t$.

The proof is similar to that for Lemma 12, except that

here b is only odd.

If $f = (1,1,1,1,4,a_6,\ldots,a_n)$, $d = 2a_h + 10$, $t = 8$, $E = m - 10$, and

$$U = 12(4mA - 9m^2 - 4m + 12), \qquad V = 64(mA + 6m + 4),$$
$$P = 4 + 2m(2a_h + 7), \qquad U + V - P^2 = 16(7mA - jm^2 + sm + 24),$$
$$j = 19 + 7a_h + a_h^2, \qquad s = 2(7 - a_h),$$
$$F = 256\{m^2A^2 - 2(79 + 49a_h + 7a_h^2)m^3A - 4(11 + 7a_h)m^2A + cm^4 + fm^3 + gm^2 + hm\},$$
$$c = 361 + 266a_h + 87a_h^2 + 14a_h^3 + a_h^4, \qquad h = -96a_h,$$
$$f = 4(29 - 30a_h + a_h^3), \qquad g = 4(1 - 98a_h - 11a_h^2).$$

Thus $F \geqq 0$ if $A \geqq 2(79 + 49a_h + 7a_h^2)m + 4(11 + 7a_h)$, a value satisfying the minor conditions also. Theorems 1 and 8 give for

$$a_5 = 5, \quad A \geqq 1264m + 160, \quad A \geqq 998m + 184,$$
$$a_5 = 7, \quad A \geqq 2132m + 216, \quad A \geqq 1530m + 240,$$

respectively.

For $f = (1,1,2,2,2,a_6,\ldots,a_n)$, $t = 8$, $d = 2a_h + 10$, and $E = m - 10$. Formulas (11), (13), (18), (19) give

$$U = 56mA - 119m^2 - 84m + 196, \qquad V = 64(mA + 6m + 4),$$
$$P = 2 + m(4a_h + 15), \qquad U + V - P^2 = 8(15mA - jm^2 + sm + 56),$$
$$j = 2a_h^2 + 15a_h + 43, \qquad s = 2(15 - a_h).$$
$$F = 64\{m^2A^2 - 2(407 + 225a_h + 30a_h^2)m^3A - 12(9 + 5a_h)m^2A + cm^4 + fm^3 + gm^2 + hm\},$$
$$c = 1849 + 1290a_h + 397a_h^2 + 60a_h^3 + 4a_h^4, \qquad h = -224a_h,$$
$$f = 4(69 - 182a_h - 15a_h^2 + 2a_h^3), \qquad g = 4(1 - 450a_h - 55a_h^2).$$

If $A \geqq 2(407 + 225a_h + 30a_h^2)m + 12(9 + 5a_h)$, $F \geqq 0$ and the minor conditions hold.

We cannot make the proof for $f = (1,1,2,2,2)$ since we

cannot guarantee b odd.

For $f = (1,1,2,2,4,a_6,\ldots,a_n)$ we need

LEMMA 18. *Let* $f = (1,1,2,2,4,a_6,\ldots,a_n)$ *where* $a_k \leqq w_{k-1} + 1$ $(6 \leqq k \leqq n)$. *Then there exist integers* a, b, r, *each* $\geqq 0$, *such that* $A = m(a - b)/2 + b + r$, $a \equiv b \pmod 2$, $b \equiv 0 \pmod 2$, F_6 *represents* $r \leqq m - 12$, *and* b *is one of* $\beta - a_h + 1, \ldots, \beta + a_h + 11$.

The proof is like that for Lemma 17 except that here b is even.

From $t = 10$, $d = 2a_h + 12$, $E = m - 12$, it follows that

$$U = 64\,(mA - 2m^2 - 2m + 4), \qquad V = 20(4mA + m^2 + 28m + 20),$$

$$P = 4 + 2m(2a_h + 9), \quad U + V - P^2 = 16(9mA - jm^2 + sm + 40),$$

$$j = 27 + 9a_h + a_h^2, \qquad s = 2(9 - a_h),$$

$$F = 256\{m^2A^2 - 2(173 + 81a_h + 9a_h^2)m^3A - 4(19 + 9a_h)m^2A + cm^4 + fm^3 + gm^2 + hm\},$$

$$c = 769 + 486a_h + 135a_h^2 + 18a_h^3 + a_h^4, \qquad h = -160a_h,$$

$$f = 4(47 - 54a_h + a_h^3), \quad g = 4(1 - 198a_h - 19a_h^2).$$

If $A \geqq 2(173 + 81a_h + 9a_h^2)m + 4(19 + 9a_h)$, $F \geqq 0$ and the minor conditions are satisfied. Theorems 1 and 8 give for

$$a_6 = 5, \quad A \geqq 4564m + 408, \quad A \geqq 1606m + 256,$$

$$a_6 = 7, \quad A \geqq 6904m + 528, \quad A \geqq 2362m + 328,$$

$$a_6 = 9, \quad A \geqq 9724m + 648, \quad A \geqq 3262m + 400,$$

$$a_6 = 11, \quad A \geqq 13024m + 768, \quad A \geqq 4306m + 472,$$

respectively.

Collecting the results, we have

THEOREM 7. *The functions* $f_1 = (1,1,1,1,a_5,\ldots,a_n)$, $a_5 = 1, 2, 3$, *satisfying* (4), *and* $f_2 = (1,1,1,2,2,a_6,\ldots,a_n)$ *satisfying* (5) *are universal*.

THEOREM 8. Let f be one of the seven sets below and let f satisfy the corresponding condition (4), (5), or (6). Then f represents every positive integer A except $15m + 6 < A < N$ where

$N = 2(79 + 49a_h + 7a_h^2)m + 4(11 + 7a_h)$ for $(1,1,1,1,4,a_6,\ldots,a_n)$,

$N = 1194m + 140$ for $(1,1,1,1,5,a_6,\ldots,a_n)$,

$N = 241.5m + 59$ for $(1,1,1,2,4,a_6,\ldots,a_n)$,

$N = 2(407 + 225a_h + 30a_h^2)m + 12(9 + 5a_h)$ for $(1,1,2,2,2,a_6,\ldots,a_n)$,

$N = 1676m + 176$ for $(1,1,2,2,4,a_6,\ldots,a_n)$,

$N = 1606m + 256$ for $(1,1,2,2,4,5,a_7,\ldots,a_n)$,

$N = 2281m + 216$ for $(1,1,2,2,5,a_6,\ldots,a_n)$.

Here the first and fourth values of f are subject to (35) and $n \neq 5$ for the fourth.

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VITA

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